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On the Solution of p-Laplacian Equation

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Dedication

To my mother, my father, my husband, and to all those who supported me in completing this research.

Asma Shnde

Declaration

I certify that this submitted for the degree of master is the result of my own research, except where otherwise acknowledge. And that this (or any part of the same) has not been submitted for a higher degree to any other university or institution.

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Abstract

The $p-$Laplace equation play an important role of mathematical modeling. In this work we present the model $p-$Laplace equation with zero Dirichlet boundary condition of the form

$$\begin{aligned} -\Delta_p u &= \frac{1}{\sigma}\frac{\partial F(x,u)}{\partial u} + \lambda a(x) \mid u \mid^{q-2} u, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{aligned} \tag{1}$$

where $\triangle_p$ denotes the $p-$Laplacian operator defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$; $p > 2$, Ω is abounded domain of $\mathbb{R}^n, (n \geq 3), 1 < q < p < \sigma < p^*$, ($p^* = \frac{np}{n-p}$ if $p < n, p^* = \infty$ if $p \geq n$), $\lambda \in \mathbb{R} \setminus \{0\}$, $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree σ, that is, $F(x, tu) = t^\sigma F(x, u)$ hold for all $(x, u) \in \overline{\Omega} \times \mathbb{R}$ and $a(x) : \Omega \longrightarrow \mathbb{R}$ are smooth functions which change sign in Ω. In general it is almost impossible to find analytical solutions of $p-$Laplace equation. Thus it is necessary to solve this equation in weak sense. In this Thesis, we derived the variational form of Equation (1) that used to find the critical points of this problem. We apply a method based on Nehari results on three submanifolds of the first Sobolev space $W_0^{1,p}$. The Nehari method form contains specific condition used to find critical points of the equation and to indicate that it is a non-trivial solution for problem (1). Further in this thesis we apply $p-$Laplacian equation in image denoising. In image processing, partial differential equations play an important role. In Total variational method see, [17], such equations arise from minimizing some energy functional (like the L_1 norm of the gradient). Other methods are designed using geometrical arguments (like evolution tangent to isophotes, known as Mean Curvature Motion [8]. In this work, a general parameter-driven framework for both approaches is given, [24], that have one specific common element, the Gaussian scale space [13]. For the first set of equation, the L_p norm of the gradient is used with p a free parameter, thus obtaining so-called $p-$Laplacians [14]. The evolution equation is a PDE that can be simplified using (geometrical) gauge coordinates. A numerical experiment related to image denoising is presented in this thesis.

Contents

Chapter 1

Introduction

The $p-$Laplacian, or the $p-$Laplace operator is a quasilinear elliptic partial differential operator of second order. The $p-$Laplacian equation is a generalization of the partial differential equation of Laplace equation. Many nonlinear problem in physics and mechanics are formulated in equations that contain the $p-$Laplacian. The study of these equations started more than thirty years ago. In the last few years, $p-$Laplacian equations have increasing attention, and rapid development has been achieved for the study of the equations involving operator Delta p. This theory has been developed very quickly and attracted a considerable interest from researches, since the $p-$Laplacian operator a rise from many applied fields such as turbulent filtration in blood flow problems and material science etc. Several problems involving $\triangle_p$ operator for Dirichlet or Neumann boundary condition have been studied by many researchers such as, Drabek et al.[19], Ambrosetti et al.[4], Brezis and Nirenberg [10], Tehrani [12] by using variational methods and Amman and Lopez-Gomez [11] by using global bifurcation theory. In recent years, several authors have used the Nehari manifold and fibering maps (i.e.,maps of the form $t \longrightarrow J_\lambda(tu)$, where J_λ is the Euler functional associated with the equation) to solve semilinear and quasilinear problems. By the fibering method, Drabek and pohozaev [20], Bozhkov and Mitidieri [26] studied, respectively, the existence of multiple solution to a $p-$Laplacian system. Brown and Zhang [16] have studied the subcritical semilinear elliptic equation with a sign-changing weight function

$$\begin{aligned} - \triangle u(x) &= \lambda a(x)u + b(x)|u|^{\gamma-2}u, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\gamma > 2$. Exploiting the relationship between the Nehari manifold and fibering maps, they gave an interesting explanation of the well known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value, the author considered above problem with $1 < \gamma < 2$. Also, the authors in [26] by the same arguments they considered the semilinear elliptic problem :

$$\begin{aligned} - \triangle u(x) &= \lambda f(x)|u|^{q-2}u(x) + g(x)|u|^{p-2}u, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{aligned} \tag{1.2}$$

where $1 < q < 2 < p$. Affected by the work of Brown and Zhang [16] treated the problem:

$$\begin{aligned} \Delta(|\Delta u|^{p-2}\Delta u) &= \frac{1}{p^*}f(x,u) + \lambda|u|^{q-2}u, \text{ in } \Omega \\ u &= \Delta u = 0, \text{ on } \partial\Omega \end{aligned} \tag{1.3}$$

where f is positively homogenous of degree $p^* - 1$. In this thesis, motivated by the above works, we give a simple variational method which is similar to the fibering method to prove the existence of at least two positive solutions of problem (1). In fact we use the decomposition of the Nehari manifold as λ vary to prove the main result. In this work we consider the model equation so called the $p-$Laplacian equation

$$\begin{aligned} -\Delta_p u &= \frac{1}{\sigma}\frac{\partial F(x,u)}{\partial u} + \lambda a(x) \mid u \mid^{q-2} u, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{aligned} \tag{1.4}$$

and its corresponding energy functional

$$J_\lambda(u) = \frac{\|u\|^p}{p} - \frac{1}{\sigma}\int_\Omega F(x,u)dx - \frac{\lambda}{q}\int_\Omega a(x)|u|^q dx. \tag{1.5}$$

We consider the problem of finding the solution of Equation (1.4) as a variational problem. That is we find the minimum of J_λ on the set of functions satisfying the condition $u = 0$ on the boundary. In many problems of mathematical physics and variational calculus it is not sufficient to deal with the classical solution, of differential equations. It is necessary to introduce the notion of weak derivatives and to work in the so called Sobolev spaces. The theory of Sobolev spaces gives the basis for studying the existence of solutions (in the weak sense) of partial differential equations. Several problems in analysis can be cast into the form of functional equations $F(u) = 0$, the solution u being sought among a class of admissible functions belonging to some Banach space E. Typically, these equations are nonlinear, for instance, if the class of admissible functions is restricted by some nonlinear constraint. A particular class of functional equation is the class of Euler -Lagrange equation $DJ(u) = 0$ for a functional J on E, which is Frechet differentiable with derivative DJ. Variational principles play an important role in mathematical physics, differential geometry, optimal control and numerical analysis. Suppose J is a Frechet differentiable functional on a Banach space E with normed dual space E^* and let $DJ : E \longrightarrow E^*$ denote the Frechet derivative of J. Then the directional (Gateaux) derivative of J at u in the direction of v is given by

$$\frac{d}{d\epsilon}E(u+\epsilon v)|_{\epsilon=0} = \langle DJ(u), v\rangle = DJ(u)v \tag{1.6}$$

For such J, we call a point $u \in E$ critical if $DJ(u) = 0$, otherwise, u is called regular. A number $\beta \in \mathbb{R}$ is a critical value of J if there exists a critical point u of J with $J(u) = \beta$, otherwise, β is called regular. Of particular interest will be relative minima of J, possibly subject to constraints. We recall that for a set $N_\lambda \subset E$ a point $u \in N$ is an absolute minimizer for J_λ on N_λ if for all $v \in N_\lambda$ there holdes $J_\lambda(v) \geq J_\lambda(u)$. The purpose of this research is to firstly discuss the problem of existence of positive solutions of Equation (1.4) from the variational viewpoint and, in particular, from the view point of the Nehari manifold, $N_\lambda = \{u \in E \setminus \{0\} : \langle DJ_\lambda(u), u\rangle = 0\}$ and secondly to present an application of the $p-$Laplace equation in the field of image Denoising. The thesis is organized as follows. In Chapter Two, we give basic concepts of functional analysis used through out the thesis. In Chapter Three, we discuss the Nehari manifold and the variational framework of Problem (1.4), and show how existence of positive solutions of Equation (1.4) are linked to properties of the manifold. In Chapter Four, the concepts of gauge coordinates, variational derivatives, and $p-$Laplacian are discussed, also it will be shown that the $p-$Laplace evolution equation can be simplified using gauge coordinates. Further in this chapter, the properties of $p-$Laplace evolution equation are discussed in relation to image filtering and a model is introduced to remove the noise of image denoising. At the end of the thesis, we present the conclusions.

Chapter 2

Mathematical Framework

In this chapter we introduce some basics of functional analysis, we used in this research, also we present a unified approach to the method of Nehari manifold for functionals that have a local minimum at 0. This method is used in chapter three to derive positive solutions to $p-$Laplacian equation. The details of this chapter are covered mainly in reference [22],[21],[18].

2.1 Basic Concepts

Differential Equations

A differential equation is an equation whose unknown is a function depending on one or more variables. We speak of partial differential equation when the function depends on many variables and the problem involves partial derivatives. The unknown function, real valued, is denoted by u and depends on the variables $x_1, x_2, ..., x_n$, that constitute the point x. We denote the partial derivative by

$$\partial_i u(x) = \partial_{x_i} u(x) = \frac{\partial u(x)}{\partial x_i}, \;\; 1 \leq i \leq n \tag{2.1}$$

Further by $D^\alpha u(x)$ we mean $D^\alpha u(x) = \dfrac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} ... \partial x_n^{\alpha_n}}$,
where $\alpha = (\alpha_1, \alpha_2, ..\alpha_n), |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ and $\alpha_i \in \mathbb{R}^+, i = 1, 2, ...n$. such α is called a multi-index.

Example 2.1.1:

Let $u = u(x, y)$ be a function of two real variables, let $\alpha = (1, 2)$. Then α is a multi-index of order 3 and $D^\alpha u = \partial_x \partial_y^2 u = u_{xyy}$.

More precisely we have the following definition of a partial differential equation:

Definition 2.1.2:

An expression of the form

$$F(D^k u(x), D^{k-1}u(x), .., Du(x), u(x), x) = 0, \quad x \in U, \tag{2.2}$$

where U is an open subset of $\mathbb{R}^n$, and $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times U \longrightarrow \mathbb{R}$ is given, $u : U \longrightarrow \mathbb{R}$ is the unknown, k is a nonnegative integer and $D^j u(x) = D^\beta u(x)$ with $|\beta| = j$ is called a partial differential equation.

Definition 2.1.3:

The partial differential equation (2.2) is called linear if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

for given functions f and it is called homogeneous if $f \equiv 0$, otherwise it is nonhomogeneous. If the partial differential equation (2.2) depends nonlinearly upon the unknown function or any of its derivative, it is called nonlinear.

Example 2.1.4:

(1) $u_t + u_x = 0$ is homogeneous linear,

(2) $u_{xx} + u_{yy} = x^2 + y^2$ is inhomogeneous linear,

(3) $u_t^2 + u_x^2 = 0$ is not linear.

(3) $u_t + u_{xxx} + uu_x = 0$ is not linear.

Definition 2.1.5:

We say a k-th order nonlinear partial differential equation is semilinear if it can be written in the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, ..., Du, u, x) = 0.$$

In particular, this means that semilinear equations are ones in which the coefficients of the terms involving the highest-order derivatives of u depend only on x, not on u or its derivatives.

Example 2.1.6:

(1) $u_t + u_x + u^2 = 0$ is semilinear,

(2) $u_{xxx} + uu_x + u_t = 0$ is semilinear,

(3) $u_t + uu_x = 0$ is not semilinear.

Definition 2.1.7:

We say a k-th order nonlinear partial differential equation, which is not semilinear, is quasilinear if it can be written in the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, ..., Du, u, x) D^\alpha u + a_0(D^{k-1}u, ..., Du, u, x) = 0$$

In particular, this means that quasilinear equations are those equations in which the coefficients of the highest order terms may depend on $x, u, ..., D^{k-1}u$, but not on $D^k u$.

Example 2.1.8:

(1) $u_t + uu_x = 0$ is quasilinear,

(2) $u_x^2 + u_y^2 = 1$ is not quasilinear.

Definition 2.1.9:

A solution to the PDE (2.2) is a function u that satisfies (2.2) and possibly satisfies certain boundary condition on the boundary of u when it is bounded.

Among the important partial differential equations are the Laplace and the $p-$Laplace (usually called $p-$laplacian) differential equations. The Laplace operator denoted by Δ given by $\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2}$. We define the potential or Laplace equation as

$$\triangle u = 0 \text{ in } \Omega \subseteq \mathbb{R}^n.$$

This equation is a second order linear PDE. Laplace equation is the prototype for linear elliptic equations. It is less well known that it also has a nonlinear counterpart, the so called $p-$Laplace equation or ($p-$harmonic equation), depending on a parameter p belongs to $(1,\infty]$. If $p \in (1,\infty)$, then the $p-$Laplacian equation is given by the divergence form $\triangle_p u := \bigtriangledown.(|\nabla u|^{p-2}\nabla u) = 0$, when $p = \infty$, the $p-$Laplacian equation is given as

$$\triangle_\infty u := \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

which is the so called $\infty-$Laplacian equation. The $p-$Laplacian equation is an elliptic partial differential equation, which is degenerate if $p > 2$, and singular at point where $\nabla u = 0$ for $1 < p < 2$. If $p = 2$, then the $p-$Laplacian equation reduces to the simpler classical linear Laplace equation $\triangle u := \nabla.\nabla u = 0$.

The connection between the $p-$Laplacian, $p \in (1,\infty)$, and the $\infty-$Laplacian equation rely on the calculation

$$\triangle_p u = \nabla.(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-4}\{|\nabla u|^2 \triangle u + (p-2)\sum_{i,j}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\} = 0.$$

Let $p \longrightarrow \infty$ and divide by $|\nabla u|^{p-4}$, then we obtain the $\infty-$Laplacian equation. In the last few years, $p-$Laplacian equation have received increasing attention. This theory has been developed very quickly and attracted a considerable interest from researches, since the $p-$Laplacian operator arise from many applied fields.

Boundary conditions.

Definition 2.1.10:

A problem is said to be well posed, if exactly one solution exists and it continuously depends on the given data.

In fact $p-$Laplacian equation is not well posed since the solution is not unique. Many PDEs arise from physical problems, where the behaviour of the unknown function can be imposed or measured on the boundary. The most commonly used boundary conditions are:

1) Dirichlet (or Essential) Boundary condition, defined as $u = g$ on $\partial\Omega$, in particular, if $g = 0$ we speak of homogeneous boundary conditions.

2) Neumann (or Natural) Boundary conditions, defined as $\dfrac{\partial u}{\partial n} = g$ on $\partial\Omega$, where n is the outward pointing unit normal vector on $\partial\Omega$.

3) Robin Boundary conditions, defined as $\gamma u + \alpha\dfrac{\partial u}{\partial n} = g$ on $\partial\Omega$, where γ and α are real numbers.

Definition 2.1.11:

A function, which satisfies a PDE as well as the associated boundary conditions is called a classical solution.

Sobolev space.

A Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p-norm of the function itself and its derivatives up to given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space. Intuitively, a Sobolev space is a space of functions with sufficiently many derivatives for some application domain, such as partial differential equations, and equipped with a norm that measures both the size and regularity of a function. The theory of Sobolev spaces introduce by Russian mathematician Sergei Sobolev around 1938. Their importance comes from the fact that solutions of partial differential equations are naturally found in Sobolev spaces, rather than in spaces of continuous functions and with the derivatives understood in the classical sense.

Definition 2.1.12: (Lebesgue Spaces)

The space of functions that are Lebesgue integrable on Ω, open and bounded in $\mathbb{R}^n$, to the power of $p \in [1, \infty)$ is denoted by

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} : f \text{ is Lebesgue measurable and } \int_\Omega |f|^p dx < \infty\}$$

which is equipped with the norm

$$\|f\|_{L^p}(\Omega) = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Definition 2.1.13:

The space $C_0^\infty(\Omega)$ space of infinitely often differentiable real functions with compact (closed and bounded) support in Ω is denoted by

$$C_0^\infty(\Omega) = \{v : v \in C^\infty(\Omega),\ \mathrm{supp}(v) \subset \Omega\},$$

where $\mathrm{supp}(v) = \overline{\{x \in \Omega : v(x) \neq 0\}}$. In particular, functions from $C_0^\infty(\Omega)$ vanish in a neighborhood of the boundary.

Weak Derivative

Suppose, as usual, that Ω is an open set in $\mathbb{R}^n$.

Definition 2.1.14:

A function $f \in L^1_{loc}(\Omega)$ is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f \partial_i \phi dx = -\int_\Omega g_i \phi dx \ \ \forall \phi \in C_c^\infty(\Omega).$$

The function g_i is called the weak ith partial derivative of f and is denoted by $\partial_i f$. Thus, for weak derivatives, the integration by part formula

$$\int_\Omega f \partial_i \phi(x) dx = -\int_\Omega \phi(x) \partial_i f dx.$$

holds by definition for all $\phi \in C_0^\infty$. Since C_c^∞ is dense in $L^1_{loc}(\Omega)$, the weak derivative of a function, if it exists, is unique up to pointwise almost everywhere equivalence. Moreover, the weak derivative of a continuously differentiable function agrees with the pointwise derivative. The existence of a weak derivative is, however, not equivalent to the existence of a pointwise derivative almost everywhere. Higher-order weak derivatives are defined in a similar way.

Definition 2.1.15:

Let $f : \Omega \longrightarrow \mathbb{R}$ be given. Then we say that $g : \Omega \longrightarrow \mathbb{R}$ is the $\alpha-$weak derivative of f for some multi-index α, if for each $\phi \in C_0^\infty(\Omega)$, the following integration by parts formula holds:

$$\int_\Omega f D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_\Omega g \phi(x) dx,$$

where $|\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$.

Remark 2.1.16:

(1) If the $\alpha-$weak derivative exists, then it is unique.

(2) If $u \in C^{|\alpha|}$, the space of all continueously diffirantiable functions up to order $|\alpha|$ then the weak and the classical derivative coincide, which is why the same symbol D^α is used.

Remark 2.1.17:

Classical derivatives are defined pointwise, as limits of difference quotients. On the other hand, weak derivatives are defined only in an integral sense, up to a set of measure zero. By arbitrarily changing the function f on a set of measure zero we do not affect its weak derivatives in any way.

Let us consider some examples of weak derivatives that illustrate the definition. We denote the weak derivative of a function of a single variable by a prime.

Example 2.1.18:

Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} x, & x \in [0,1], \\ 1, & x \in [1,2]. \end{cases}$$

Then, for any function $\phi : [0,2] \longrightarrow \mathbb{R}$ differentiable with $\phi(0) = \phi(2) = 0$, we have that

$$-\int_0^2 f(x)\phi^{'}(x)dx = -\int_0^1 f(x)\phi^{'}(x)dx - \int_1^2 \phi^{'}(x)dx,$$

working with the first term in the right-hand side, we use integration by parts to get

$$-\int_0^1 x\phi^{'}(x)dx = -x\phi(x)|_0^1 + \int_0^1 \phi(x)dx = -\phi(1) + \int_0^1 x\phi(x)dx.$$

The fundamental theorem of calculus plus the assumption that $\phi(2) = 0$ on the second term on the right-hand side gives $-\int_1^2 \phi^{'}(x)dx = -\phi(2) + \phi(1) = \phi(1)$. We have that

$$-\int_0^2 f(x)\phi^{'}(x)dx = \int_0^1 \phi(x)dx = \int_0^2 g(x)\phi(x)dx,$$

where g is given by

$$g(x) = \begin{cases} 1, & x \in [0,1), \\ 0, & x \in [1,2]. \end{cases}$$

Hence $g = f^{'}$ is a weak derivatives of f.

Example 2.1.19:

Consider the function $u(x) = |x|$ defined on $(-1, 1)$. For $\phi \in C_0^\infty(-1, 1)$ we have

$$\begin{aligned} -\int_{-1}^{1} u(x)\phi^{'}(x)dx &= -\int_{-1}^{0}(-x)\phi^{'}(x)dx - \int_{0}^{1} x\phi^{'}(x)dx \\ &= -\int_{0}^{1} x\phi^{'}(x)dx + \int_{-1}^{0} x\phi^{'}(x)dx, \end{aligned}$$

by using integration by parts and the fact ϕ is zero at end points we obtained $-\int_0^1 x\phi'(x)dx + \int_{-1}^0 x\phi'(x)dx = \int_0^1 \phi(x)dx - \phi(1).1 + \phi(0).0 - \int_{-1}^0 \phi(x)dx + \phi(0).0 + \phi(-1).1 = \int_0^1 x\phi(x)dx - \int_{-1}^0 \phi(x)dx = \int_{-1}^1 \phi(x)v(x)dx$,

where

$$v(x) = \begin{cases} 1, & x \in (0, 1], \\ -1, & x \in [-1, 0). \end{cases}$$

Thus $v = u^{'}$ is the weak derivative of u. Note that it is not defined at $x = 0$. In fact weak derivatives are generally only defined a.e (i.e defined except on a set of measure zero), but this does not matter since we always integrate them against another function.

Example 2.1.20:

Consider the function

$$f(x) = \begin{cases} 0, & x \text{ is rational} \\ 2 + \sin x, & x \text{ is irrational.} \end{cases}$$

Clearly f is discontinous at every point x. Hence it is not differentiable at any point. On the other hand, the function $g(x) = \cos x$ provides a weak derivative for f, see [1].

Example 2.1.21:

The discontinuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

is not weakly differentiable. To prove this, note that for any test function ϕ,

$$\int_\Omega f\phi^{'}dx = \int_0^\infty \phi^{'}(x)dx = -\phi(0).$$

Thus the weak derivative $g = f'$ would have to satisfy

$$\int_{\Omega} g\phi(x)dx = \phi(0) \; \forall \phi \in C_0^{\infty}. \tag{2.3}$$

Assume for contradiction that $g \in L^1_{loc}(\mathbb{R})$ satisfy (2.3). By considering test functions with $\phi(0) = 0$, we see that g is equal to zero pointwise almost everywhere, and then (2.3) does not hold for test functions with $\phi(0) \neq 0$.

The pointwise derivative of the discontinuous function f in the previous example exists and is zero except at 0, where the function is discontinuous, but the function is not weakly differentiable.

Example 2.1.22:

Define $f \in C(\mathbb{R})$ by

$$f(x) = \begin{cases} x, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then f is weakly differentiable, with $f' = \chi_{[0,\infty)}$, where $\chi_{[0,\infty)}$ is the step function

$$\chi_{[0,\infty)}(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Definition 2.1.23:

For $k = 1, 2, 3....n$ and $p \in [1, \infty)$, we define the Sobolev space $W^{k,p}(\Omega)$ as

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^{\alpha}u \in L^p(\Omega), 0 \leq |\alpha| \leq k\}.$$

Further, we set

$$W_0^{k,p}(\Omega) = \text{ the closure of } C_0^{\infty}(\Omega) \text{ in } W^{k,p}(\Omega).$$

These spaces are equipped with the following norms

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{0 \leq |\alpha| \leq k} \|D^{\alpha}(u)\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \text{ if } 1 \leq p \leq \infty,$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{0 \leq |\alpha| \leq k} \|D^{\alpha}(u)\|_{L^{\infty}(\Omega)}.$$

Theorem 2.1.24:

The Sobolev space $W^{k,p}$ with the norm $\|\|_{W^{k,p}}$ is a complete normed vector space and thus a Banach space.

Definition 2.1.25:

(A) A function $u : \Omega \longrightarrow \mathbb{R}$ is called Lipschitz continuous if $|u(x) - u(y)| \leq L|x - y|$, where L is a positive real number.

(B) The domain Ω has a Lipschitz boundary (or Ω is a Lipschitz-domain), if for $m \in N$ there exists some open sets $U_1, U_2.., U_m \subset \mathbb{R}^n$ such that

(1) $\partial\Omega \subset \cup_{i=1}^{m} U_i$

(2) $\partial\Omega \cap U_i$ can be described as graph of a Lipschitz continous function for every $1 \leq i \leq m$.

Theorem 2.1.26: (Trace Theorem)

Let $\Omega \subset \mathbb{R}^n$ be open bounded and $\partial\Omega$ is C^1. Then there is exactly one linear and continuous operator $T : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$, $p \in [1, \infty)$ which gives for functions $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, the classical boundary values $Tu(x) = u(x)$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ i.e $Tu(x) = u(x) \mid_{x \in \partial\Omega}$

Remark 2.1.27:

On the trace

(i) the operator T is called trace or trace operator.

(ii) since a linear and continuous operator is bounded, there is a constant $C > 0$ with $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for all $u \in W^{1,p}$

From the Trace Theorem we can derive a very useful definition when dealing with homogeneous Dirichlet boundary conditions.

Definition 2.1.28:

We define the Sobolev space with functions vanishing at the boundary as

$$W_0^{k,p} = \{u \in W^{k,p}(\Omega) : u|_{\partial\Omega} = 0\}.$$

In particular, for $k = 1$ and $p = 2$ it follows that

$$W_0^{1,2} = H_0^1 = \{u \in H^1 : u|_{\partial\Omega} = 0\}.$$

The difference between $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ is not merely a technical one. The idea of the space $W_0^{k,p}(\Omega)$ is that it consists of those functions in $W^{1,p}(\Omega)$ which take the value zero at the boundary of Ω. Now many boundary value problems are equivalent to

$$Au = 0 \tag{2.4}$$

where $A : X \longrightarrow Y$ is a mapping between two Banach spaces. When the problem is variational, there exists a differentiable functional $\phi : X \longrightarrow \mathbb{R}$ such that $A = \phi'$, i.e

$$\langle Au, v\rangle = \lim_{t \longrightarrow 0} \frac{\phi(u + tv) - \phi(u)}{t}. \tag{2.5}$$

The space Y corresponds then to the topological dual X^* of X and equation (2.4) is equivalent to $\phi'(u) = 0$, i.e

$$\langle \phi'(u), v\rangle = 0, \ \forall v \in X \tag{2.6}$$

A critical point of ϕ is a solution u of (2.6) and the value of ϕ at u is a critical value of ϕ. How to find critical values? When ϕ is bounded from below, the infimum

$$c = \inf_X \phi \tag{2.7}$$

is a natural candidate. Ekelands variational principle implies the existence of a sequence (u_n) such that

$$\phi(u_n) \longrightarrow c, \ \phi'(u_n) \longrightarrow 0, \ \text{as } n \longleftrightarrow \infty. \tag{2.8}$$

Such a sequence is called a Palais Smale sequence at level c. The functional ϕ satisfies the $(PS)_c$ condition if any Palais-Smale sequence at level c has a convergent subsequence.

If ϕ is bounded from below and satisfies the $(PS)_c$ condition at level $c := \inf_X \phi$, then c is a critical value of ϕ.

Definition 2.1.29: (Gateaux Derivative)

Let $\phi : U \longrightarrow \mathbb{R}$ where U is an open subset of a Banach space E. The functional ϕ has a Gateaux derivative $\phi^{'} \in E^*$ at $u \in U$ if, for every $h \in E$,

$$d_h\phi = \langle \phi^{'}(u), h \rangle = \lim_{t \longrightarrow 0} \frac{\phi(u + th) - \phi(u)}{t}$$

the Gateaux derivative at u is denoted by $\phi'(u)$. The functional ϕ has a Frechet derivative $f \in E^*$ at $u \in U$ if

$$\langle \phi^{'}(u), h \rangle = \lim_{h \longrightarrow 0} \frac{1}{\|h\|}(\phi(u + h) - \phi(u))$$

the functional $\phi \in C^1(U, \mathbb{R})$ if the Frechet derivative of ϕ exists and is continuous on U.

Example 2.1.30:

Let $J : H^1(\Omega) \longrightarrow \mathbb{R}$ be a functional defined by $J = \displaystyle\int_\Omega \frac{1}{2}u_x^2 + \frac{1}{2}u^2 dx$.

Then the Gateaux derivative

$$d_hJ = \lim_{\epsilon \longrightarrow 0} \frac{\int_\Omega \left[\frac{1}{2}u_x^2 + \frac{1}{2}u^2 + \epsilon uh + \epsilon u_x h_x + \frac{1}{2}\epsilon^2 h_x^2 + \frac{1}{2}\epsilon h^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u^2\right] dx}{\epsilon}.$$

Therefore $d_hJ = \displaystyle\int_\Omega (uh + u_x h_x)dx$

Definition 2.1.31:

A critical, or stationary point of $J_\lambda : E \longrightarrow \mathbb{R}$ is a $z \in E$ such that J_λ is differentiable at z and $DJ_\lambda(z) = 0$. A critical level of J_λ is a number $c \in \mathbb{R}$ such that there exists a critical points $z \in E$ with $J_\lambda(z) = c$.

2.2 Some Basic Lemmas

This section contains basic lemmas and theorems without proof that are needed later in the research. For the proof of these theorems and lemmas see [22],[18].

Theorem 2.2.1: (Lebesgue Dominated Convergence)

Suppose $f_n : \mathbb{R} \longrightarrow \mathbb{R}^*$ are lebesgue measurable function such that $f(x) = \lim_{n\to\infty} f_n(x)$ exist, assume there exist integrable $g : \mathbb{R} \longrightarrow [0, \infty)$ with $|f_n(x)| \leq g(x), \forall x \in \mathbb{R}$, then f is integrable as is f_n for each n and

$$\lim_{n\to\infty} \int_R f_n d\mu = \int_R f d\mu$$

Definition 2.2.2: (Holder Inequality)

Let $\frac{1}{p} + \frac{1}{q} = 1,\ p, q \in [1, \infty)$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and it holdes that

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Remark 2.2.3:

Let us recall that $J \in C(E, \mathbb{R})$ is coercive if $\lim_{\|u\|\to\infty} J(u) = \infty$.

Remark 2.2.4:

J_λ is called weakly lower semi continuous if for every sequence $u_n \rightharpoonup u$ one has that $J_\lambda(u) \leq \lim \inf_{n\longrightarrow\infty} J_\lambda(u_n)$.

Lemma 2.2.5:

Let E be a reflexive Banach space and let $J_\lambda : E \longrightarrow \mathbb{R}$ be coercive and weakly lower semi continuous. Then J_λ is bounded from below on E, and there exists $c \in \mathbb{R}$ such that $J_\lambda(u) \geq c$ for all $u \in E$.

Theorem 2.2.6:

Every bounded sequence of finite measures on Ω contains a weakly convergent subsequence. If $u_n \longrightarrow u$ in $M(\Omega)$ then u_n is bounded and

$$\|u\| \leq \underline{\lim}\|u_n\|$$

where $M(\Omega)$ denote the space of finite measures.

Lemma 2.2.7: (Brezis-Lieb Lemma,1983)

Let Ω be an open subset of $\mathbb{R}^N$ and let $(u_n) \subset L^p(\Omega), 1 \leq p < \infty$. If

a (u_n) is bounded in $L^p(\Omega)$,

b $u_n \longrightarrow u$ almost everywhere on Ω,then

$$\lim_{n \longrightarrow \infty} (|u_n|_p^p - |u_n - u|_p^p) = |u|_p^p.$$

Theorem 2.2.8: (Fatous lemma)

Let $A \subset \mathbb{R}^n$ be measurable and let f_n be a sequence of nonnegative, measurable functions. Then

$$\textstyle\int_A \left(\lim_{n \longrightarrow \infty} \inf f_n(x) \right) dx \leq \lim_{n \longrightarrow \infty} \inf \displaystyle\int_A f_n(x) dx.$$

Theorem 2.2.9:

let Ω be an open subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$, if $v_n \longrightarrow u$ in L^p, there exists a subsequence w_n of v_n and $g(x) \in L^p$ such that, $w_n \longrightarrow u$ a.e on Ω and $|u| \leq g(x)$, $|w_n| \leq g(x)$.

Theorem 2.2.10: (Ekelands variational principle)

Let X be a Banach space, $\phi \in C^1(X, \mathbb{R})$ bounded below, $v \in X$ and $\epsilon, \delta > 0$. If $\phi(v) \leq \inf_X \phi + \epsilon$ there exists $u \in X$ such that $\phi(u) \leq \inf_X \phi + 2\epsilon$, $\|\phi'(u)\| < \dfrac{8\epsilon}{\delta}$, $\|u - v\| \leq 2\delta$.

Theorem 2.2.11:

Let $\phi \in C^1(X, \mathbb{R})$ be bounded below. If ϕ satisfies condition (PS)c with $c = \inf_X \phi$ then every minimizing sequence for ϕ contains a converging subsequence. In particular, there exists a minimizer for ϕ contains a converging subsequence. In particular, there exists a minimizer for ϕ.

Theorem 2.2.12: (Rellich-Kondrachov Lemma)

On a bounded open set Ω, the nonendpoint Sobolev Embeddings

$$W_0^{1,p} \longrightarrow L^q(\Omega),$$

where $q < \frac{np}{n-p} = p^*$ is compact.

2.3 Abstract Setting for Nehari Manifold

In 1960, [2], Nehari has introduced a method which turned out to be very useful in critical point theory and eventually came to bear his name. He considered a boundary value problem for a certain nonlinear second order ordinary differential equation in an interval (a,b) and showed that it has a nontrivial solution which may be obtained by constrained minimization of the Euler-Lagrange functional corresponding to the problem. In 1961, he proved the existence of infinitely many solution and, in 1963 he solved the case where $\Omega = \mathbb{R}^3$. To describe Neharis method, let E be real Banach space and $\phi \in C^1(E, \mathbb{R})$ a functional. The Frechet derivative of ϕ at u, $\phi'(u)$ is an element of the dual space E^*, and we shall denote $\phi'(u)$ evaluated at $v \in E$ by $\langle \phi'(u), v \rangle$. Suppose $u \neq 0$ is a critical point of ϕ, i.e. $\phi'(u) = 0$. Then necessarily u is contained in the set

$$N = \{u \in E \backslash \{0\} : \langle \phi^{'}(u), u \rangle = 0\}. \tag{2.9}$$

So N is a natural constraint for the problem of finding nontrivial (i.e.,$\neq 0$) critical points of ϕ. N is called the Nehari manifold though in general it may not be a manifold. Set

$$c := \inf_{u \in N} \phi(u). \tag{2.10}$$

Under appropriate conditions on ϕ one hopes that c is attained at some $u_0 \in N$ and that u_0 is a critical point. Assume without loss of generality that $\phi(0) = 0$. Assume that for each $w \in S_1(0) := \{w \in E : \|w\| = 1\}$ the function $\alpha_w(s) = \phi(sw)$ attains a unique maximum s_w in $(0, \infty)$ such that $\alpha^{'}_w(s) > 0$ whenever $0 < s < s_w, \alpha^{'}_w(s) < 0$ whenever $s > s_w$ and $s_w \geq \delta$ for some $\delta > 0$ independent of $w \in S_1(0)$. Then $\alpha^{'}_w(s_w) = \phi^{'}(s_w w)w = 0$. Hence $s_w w$ is the unique point on the ray $s \longrightarrow sw, s > 0$, which intersects N. Moreover N is bounded away from 0. It is easy to see that N is closed in E and there exists a radial bijection between N and $S_1(0)$. It is proved that if s_w is bounded on compact subsets of $S_1(0)$, then this bijection is in fact a homeomorphism. Clearly, c in (2.10), if attained, is positive. Further it is shown that $u_0 \in N$ is a critical point whenever $\phi(u_0) = c$. Note that since $s \longrightarrow \alpha_w(s)$ is increasing for all $w \in S_1(0)$ and $0 < s < \delta_0$, is a local minimum and hence a critical point of ϕ. Since u_0 is a solution to the equation $\phi^{'}(u) = 0$ which has minimal energy ϕ in the set of all nontrivial solutions, we shall call it a ground

state. Suppose in addition to the assumptions already made that E is a Hilbert space and $\phi \in C^2(E, \mathbb{R})$. Then
$\alpha_w^{''}(s_w) = \phi^{''}(s_w w)(w, w) = s_w^{-2}\phi^{''}(u)(u, u) \leq 0$, where $u = s_w w \in N$. If $\phi^{''}(u)(u, u) < 0$ for all $u \in N$, then setting $G(u) = \phi^{'}(u)u$, then

$$G^{'}(u)u = \phi^{''}(u)(u, u) + \phi^{'}(u)u = \phi^{''}(u)(u, u) < 0, u \in N$$

Since $N = \{u \in E\backslash\{0\} : G(u) = 0\}$, it follows from the implicit function theorem that N is a C^1-manifold of codimension 1 and $E = T_u(N) \oplus Ru$ for each $u \in N$. Hence in this case it is easily seen that any $u \in N$ with $\phi(u) = c$ (i.e., any minimizer of $\phi|_N$) satisfes $\phi^{'}(u) = 0$. More generally, a point $u \in E$ is a nonzero critical point of ϕ if and only if $u \in N$ and u is critical for the restriction of ϕ to N. In view of this property, one may apply critical point theory on the manifold N in order to find critical points of ϕ. Our goal in this research is to present a method of Nehari manifold and to introduce it can be applied to solve elleptic p-laplacian equation in problem (1). In [2] the researchers from them A.Ambrosetti, A.Malchiodi, and Nehari introduced several examples where it can be applied in order to show the existence of solutions to nonlinear boundary value problems.

Chapter 3

Positive Solution of p-Laplacian Equation with Dirichlet Boundary Conditions

Problems involving the $p-$Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids: pseudo-plastic fluids correspond to $p \in (1,2)$ while dilatant fluids correspond to $p > 2$. The case $p = 2$ expresses Newtonian fluids [7]. In this chapter we are concerned with the existence and multiplicity of positive solutions to the nonlinear elliptic problem:

$$\begin{aligned} -\Delta_p u &= \frac{1}{\sigma}\frac{\partial F(x,u)}{\partial u} + \lambda a(x) \mid u \mid^{q-2} u, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{aligned} \tag{3.1}$$

where $\triangle_p$ denotes the $p-$Laplacian operator defined by $\triangle_p = div(|\nabla u|^{p-2}\nabla u)$; $p > 2$, Ω is abounded domain of $\mathbb{R}^n, (n \geq 3), 1 < q < p < \sigma < p^*$, ($p^* = \frac{np}{n-p}$ if $p < n, p^* = \infty$ if $p \geq n$), $\lambda \in \mathbb{R} \setminus \{0\}$, $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree σ, that is, $F(x,tu) = t^\sigma F(x,u)$ hold for all $(x,u) \in \overline{\Omega} \times \mathbb{R}$ and $a(x) : \Omega \longrightarrow \mathbb{R}$ are smooth functions which change sign in Ω. Problem (3.1) is posed in the frame work of the Sobolev space $W_0^{1,p}(\Omega)$ accompained with the standard norm $\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$. In this reasearch, under the following conditions are assumed to be hold, we prove that using Nehari method equation (3.1) has two positive solutions.

1) $a(x) \in C(\Omega)$ with $\|a\|_\infty = 1, a^+ = max(+a, 0) \not\equiv 0$, $a^- = max(-a, 0) \not\equiv 0$.

2) $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $F(x, tu) = t^\sigma F(x, u)$ $(t > 0), \forall x \in \overline{\Omega}, u \in \mathbb{R}$.

3) $F(x, 0) = \frac{\partial F(x,0)}{\partial u} = 0, F^+(x, u) = max(+F(x, u), 0) \not\equiv 0$, and

$F^-(x, u) = max(-F(x, u), 0) \not\equiv 0$ $\forall u \neq 0$.

The details of this chapter are covered mainly in reference [9]. The function F satisfies the following properties.

Property 1: $u \frac{\partial F(x,u)}{\partial u} = \sigma F(x, u)$.

Proof. By Assumption (2), $F(x, tu) = t^\sigma F(x, u)$. Setting $z = tu$, and applying the chain rule we get $\dfrac{\partial F}{\partial t} = \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial t}$ or $\dfrac{\partial F(x, tu)}{\partial t} = \dfrac{\partial F(x, z)}{\partial z}\, u$. At $t = 1, z = u$ and $\dfrac{\partial F(x, tu)}{\partial t}\,|_{t=1} = \dfrac{\partial F(x, u)}{\partial u} u$. Since $\dfrac{\partial F(x, tu)}{\partial t} = \sigma\, t^{\sigma-1} F(x, u)$, we obtain $\dfrac{\partial F(x, tu)}{\partial t}\,|_{t=1} = \sigma F(x, u)$. Therefore $u\,\dfrac{\partial F(x, u)}{\partial u} = \sigma\, F(x, u)$.

Property 2: $|F(x, u)| \leq K|u|^\sigma$, for some positive constant K.

Proof. From the first property we have $u\dfrac{\partial F(x, u)}{\partial u} = \sigma F(x, u)$, $\dfrac{\frac{\partial F(x, u)}{\partial u}}{F(x, u)} = \dfrac{\sigma}{u}$. If we integrate with respect to u we get

$$\begin{aligned} ln|F(x, u)| &= \sigma\, ln|u| + k(x) \\ &= ln|u|^\sigma + k(x) \end{aligned}$$

or $|F(x, u)| = e^{k(x)}|u|^\sigma$. By continuity of $e^{k(x)}$ on $\overline{\Omega}$ then there exist $K > 0$ such that $e^{k(x)} \leq K$. Hence $|F(x, u)| \leq K|u|^\sigma,\ K > 0$.

In this chapter, firstly we study the existence and multiplicity of nontrivial solutions of the $p-$laplacian equation with zero Dirichlet boundary conditions. In Section One we discuss the relation between the weak solution of equation (3.1) and variational form, we also present some technical lemmas which are useful in the proof of main result Theorem (3.2.1). Finally in Section Two we introduce the proof of the Theorem (3.2.1).

3.1 Variational Form of Differential Equations

The modern study is often based on the weak form of a partial differential equation, as too are various numerical solution techniques for finding approximate solutions. The weak form of a partial differential equation is empowering for mathematical analysis as tools from functional analysis can be leveraged. Weak formulations are often referred to as variational formulations, but they can still be formulated for problems that cannot be phrased as a minimization problem. Classical transport equations are a typical example of a case that cannot be posed as a minimization problem. The derivation of the weak form of a differential equation follows a standard process:

1. Multiply the differential equation by an arbitrary weight function and integrate over the domain.

2. Apply integration by parts, if possible, and insert Dirichlet boundary condition.

The weak form of an equation does not generally make an equation easier to solve analytically (it may make it harder), but is usually a more suitable form for mathematical analysis (allowing us to say things about the properties of the equation without knowing the solution) and for numerical solution methods. To derive the weak form of Equation (3.1), we first multiply both sides of equation (3.1) by a weight function ϕ and integrate over the domain Ω

$$\int_{\Omega} -\triangle_p u \; \phi dx = \frac{1}{\sigma}\int_{\Omega} \frac{\partial F(x,u)}{\partial u} \; \phi dx - \lambda \int_{\Omega} a|u|^{q-2} u \; \phi dx, \tag{3.2}$$

we require that $\phi = 0$ on parts of the boundary. Integrating the left side by parts, we get

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi dx - \frac{1}{\sigma}\int_{\Omega} \frac{\partial F(x,u)}{\partial u} \phi dx - \lambda \int_{\Omega} a|u|^{q-2} u \phi dx = 0, \; \forall \phi \in E \tag{3.3}$$

where $E = W_0^{1,p}$, solving equation (3.1) now involves finding u that satisfies the Dirichlet boundary conditions such that the above equation holds for all functions ϕ in E. Problem (3.1) has a variational structure equivalent to the weak form (3.3).

The variational form of a differential equation is an alternative way of expressing the same problem. The variational view, and the associated machinery of variational methods and functional analysis are at the heart of the modern study of partial differential equations and provide the basis for a variety of numerical solution procedures, like the Finite Element Method. We will see that classical variational methods involve the minimization of a functional, although many of the concepts of variational methods extend beyond this classical perspective, say J_λ that depends on the function $u(x)$. We will usually want to find the function u that minimizes J_λ (sometimes we will be satisfied with stationary points). The problem is stated as

$$\min_{u\in E} J_\lambda(u)$$

The solution u is sometimes referred to as a minimizer of J_λ. In general, some constraints will be applied to u. To find u that minimizes J_λ, we take the directional derivative of J_λ and set it equal to zero,

$$DJ_\lambda(u)(\phi) = \frac{d}{d\epsilon} J(u+\epsilon\phi)|_{\epsilon=0} = 0.$$

Recall that the directional derivative is the change in J_λ if we move a small distance from u in the direction of ϕ (hence the name variational methods). For simple problems, we can apply partial differentiation directly without going through the formalities of the directional derivative. The precise definition of J_λ depends on the problem considered. The problem (3.1) has a variational structure. To explain the relation of problem (3.1) to variational problems we define the functional (energy functional) $J_\lambda : W_0^{1,p} \longrightarrow \mathbb{R}$ by

$$J_\lambda(u) = \frac{\|u\|^p}{p} - \frac{1}{\sigma}\int_\Omega F(x,u)dx - \frac{\lambda}{q}\int_\Omega a(x)|u|^q dx \tag{3.4}$$

Then we consider the following problem. Find $u \in W_0^{1,p}$such that

$$J_\lambda(u) \leq J_\lambda(\phi)\ \forall\phi \in E. \tag{3.5}$$

For such problems a necessary condition for optimality is the first variation $\delta J_\lambda(u,\phi)$ must vanish for arbitrarily admissible functions ϕ. It is defined by $\delta J_\lambda(u,\phi) = \frac{d}{d\epsilon}J_\lambda(u+\epsilon\ \phi)|_{\epsilon=0}$, such that $\delta J_\lambda(u,\phi) = DJ_\lambda(u)\phi$. For the functional $J_\lambda(u)$ defined in (3.4) we have

$$\begin{aligned} J_\lambda(u+\epsilon\phi) &= \frac{\|u+\epsilon\phi\|^p}{p} - \frac{1}{\sigma}\int_\Omega F(x,u+\epsilon\phi)dx - \frac{\lambda}{q}\int_\Omega a(x)|u+\epsilon\phi|^q dx \\ &= \frac{1}{p}\int_\Omega |\nabla(u+\epsilon\phi)|^p dx - \frac{1}{\sigma}\int_\Omega F(x,u+\epsilon\phi)dx - \frac{\lambda}{q}\int_\Omega a(x)|u+\epsilon\phi|^q dx. \end{aligned}$$

Differentiate with respect to ϵ we end with $\frac{d}{d\epsilon} J_\lambda(u+\epsilon\phi)|_{\epsilon=0} =$

$$\frac{1}{p}\int_\Omega p|\nabla u+\epsilon\nabla\phi)|^{p-2}\nabla u\nabla\phi dx - \frac{1}{\sigma}\int_\Omega \frac{\partial F(x,u+\epsilon\phi)}{\partial u}\phi dx - \frac{\lambda}{q}\int_\Omega qa(x)|u+\epsilon\phi|^{q-2}u\phi dx$$

Hence, the first variation reads

$$\begin{aligned} \delta J_\lambda(u,\phi) &= \frac{d}{d\epsilon} J_\lambda(u+\epsilon\phi)|_{\epsilon=0} \\ &= \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi dx - \frac{1}{\sigma}\int_\Omega \frac{\partial F(x,u)}{\partial u}\phi dx - \lambda\int_\Omega a(x)|u|^{q-2}u\phi dx. \end{aligned}$$

Therefore the condition $\delta J_\lambda(u,\phi) = 0$ necessary for optimality in (3.5) is equivalent to the variational form coresponding to Equation (3.1). Hence the nontrivial weak solutions are equivalent to the nonzero critical (stationary) points of the functional $J_\lambda(u)$.

In order to prove that the functional $J_\lambda(u)$ is C^1 we need the following lemma.

Lemma 3.1.1:

Assume that $F \in C^1(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ is positively homogenous of degree σ, then $\frac{\partial F}{\partial u} \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ is positively homogenous of degree $\sigma-1$.

Proof. By assumption $F(x,tu) = t^\sigma F(x,u)$. If we differentiate with respect to u we obtain

$$\frac{\partial F(x,tu)}{\partial u}t = t^\sigma\frac{\partial F(x,u)}{\partial u}$$

or

$$\frac{\partial F(x,tu)}{\partial u} = t^{\sigma-1}\frac{\partial F(x,u)}{\partial u}.$$

Hence $\frac{\partial F}{\partial u}$ is positively homogenous of degree $\sigma-1$. Since $F \in C^1(\overline{\Omega}\times\mathbb{R},\mathbb{R})$, then $\frac{\partial F}{\partial u}$ is a real valued continous function on $\overline{\Omega}\times\mathbb{R}$.

Remark 3.1.2:

There exists a positive constant K such that $\left|\dfrac{\partial F(x,u)}{\partial u}\right| \leq K|u|^{\sigma-1}$.

Proof. Using the result of the last Lemma and differentiating the function $\dfrac{\partial F(x,tu)}{\partial u}$ with respect to t then setting $t=1$ we get

$$u\,\frac{\partial^2 F(x,u)}{\partial u^2} = (\sigma-1)\,\frac{\partial F(x,u)}{\partial u}$$

or we write this as

$$\frac{\dfrac{\partial^2 F(x,u)}{\partial u^2}}{\dfrac{\partial F(x,u)}{\partial u}} = \frac{\sigma-1}{u}.$$

Integrate with respect to u to get

$$\begin{aligned} ln\left|\frac{\partial F(x,u)}{\partial u}\right| &= (\sigma-1)\; ln|u| + k(x) \\ &= ln|u|^{\sigma-1} + k(x), \end{aligned}$$

taking the exponential to both sides leads to $\left|\dfrac{\partial F(x,u)}{\partial u}\right| = e^{k(x)}|u|^{\sigma-1}$, again by continuity of $e^{k(x)}$ on $\overline{\Omega}$ then there exist $K>0$ such that $e^{k(x)} \leq K$.

Hence $\left|\dfrac{\partial F(x,u)}{\partial u}\right| \leq K|u|^{\sigma-1},\ K>0.$

Remark 3.1.3:

Let S_l denote the best Sobolev constant for the operators $W_0^{1,p}(\Omega) \longrightarrow L^l(\Omega)$, given by $S_l = \inf\limits_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \dfrac{\int_\Omega |\nabla u|^p dx}{(\int_\Omega |u|^l dx)^{\frac{p}{l}}}$, where $1<l\leq p^*$. Then

$$\int_\Omega |u|^l dx \leq S_l^{\frac{-l}{p}}\, \|u\|^l \;\forall u \in W_0^{1,p}(\Omega).$$

Proof. By definition of infimum we have $S_l \leq \dfrac{\int_\Omega |\nabla u|^p dx}{(\int_\Omega |u|^l dx)^{\frac{p}{l}}}$ or we write $S_l^{\frac{l}{p}} \leq \dfrac{(\int_\Omega |\nabla u|^p dx)^{\frac{l}{p}}}{\int_\Omega |u|^l dx}$. Therefore $\displaystyle\int_\Omega |u|^l dx \leq S_l^{\frac{-l}{p}}\, \|u\|^l$, where $\|u\| = (\displaystyle\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}$.

Lemma 3.1.4:

Let $p, r \in [1, \infty)$ and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ such that

$$|f(x, u)| \leq c(1 + |u|^{\frac{p}{r}}), \forall x \in \overline{\Omega}, \forall u \in \mathbb{R}. \tag{3.6}$$

Then for every $u \in L^p(\Omega)$, one has $f(., u) \in L^r(\Omega)$, and the operator $A : L^p(\Omega) \longrightarrow L^r(\Omega)$ defined by $A(u)(x) = f(x, u(x))$ is continuous.

Proof.

1) To prove that $f(., u) \in L^r(\Omega)$ we need to show that $\forall x \in \Omega, \displaystyle\int_\Omega |f(x, u)|^r dx < \infty$. Let $u \in L^p(\Omega)$. Since $|f(x, u)| \leq c(1 + |u|^{\frac{p}{r}})$ leads to

$$|f(x, u)|^r \leq c^r(1 + |u|^{\frac{p}{r}})^r.$$

It follows from the inequality $\|f + g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$, where $f, g \in L^p$ that

$$\begin{aligned}
\int_\Omega |c^r(1 + |u|^{\frac{p}{r}})^r| &= c^r \int_\Omega |1 + |u|^{\frac{p}{r}}|^r \\
&\leq c^r 2^{r-1} \left(\int_\Omega |1|^r dx + \int_\Omega |u|^{\frac{p}{r} r} dx \right) \\
&= 2^{r-1} c^r \left(\int_\Omega (1 + |u|^p) dx \right) < \infty.
\end{aligned}$$

Therfore $c^r(1 + |u|^{\frac{p}{r}})^r \in L^1(\Omega)$, thus $\int_\Omega |f(x, u)|^r dx < \infty$ and $f(., u) \in L^r(\Omega)$.

2) To show that $A(u)(x) = f(x, u(x))$ is continuous we need to prove that if $u_n \longrightarrow u$ in L^p then $A(u_n) \longrightarrow A(u)$ in L^r. Assume that $u_n \longrightarrow u$ in L^p. By Theorem (2.2.9) there exists a function $g(x)$ in L^p and a subsequence w_n of u_n such that $w_n \longrightarrow u$ a.e in Ω and $|u| \leq g(x)$, $|w_n| \leq g(x)$ on Ω. Then

$$\begin{aligned}
|f(x, w_n) - f(x, u)|^r &\leq (|f(x, w_n)| + |f(x, u)|)^r \\
&\leq \left(c(1 + |w_n|^{\frac{p}{r}}) + c(1 + |u|^{\frac{p}{r}}) \right)^r \\
&\leq \left(2c(1 + |g(x)|^{\frac{p}{r}}) \right)^r \\
&\leq 2^r c^r (1 + |g(x)|^{\frac{p}{r}})^r.
\end{aligned}$$

Analogus to the proof of part one we get $2^r c^r(1 + |g(x)|^{\frac{p}{r}})^r \in L^1(\Omega)$. It follows from the Dominated Convergence Theorem that $\displaystyle\lim_{n\to\infty} \int_\Omega |f(x, w_n) - f(x, u)|^r dx = 0$ which leads to $\left(\lim_{n\longrightarrow\infty}\int_\Omega |f(x,w_n)-f(x,u)|^r dx\right)^{\frac{1}{r}}=0$. Hence $\|A(w_n)-A(u)\|_L^r\longrightarrow 0$ as $n\longrightarrow\infty$ and $A(w_n)\longrightarrow A(u)$ in L^r. Thus $A(u)$ is continuous.

Remark 3.1.5:

The Gateaux derivative of the functional $J_\lambda(u)$ is given by

$$\langle J'_\lambda(u),h\rangle=\lim_{t\to 0}\frac{J_\lambda(u+th)-J_\lambda(u)}{t},$$

and if J_λ has a continuous Gateaux derivative on E then $J_\lambda\in C^1(E,\mathbb{R})$.

Lemma 3.1.6:

Suppose that $\dfrac{\partial F(x,u)}{\partial u}\in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ and $\left|\dfrac{\partial F(x,u)}{\partial u}\right|\le K|u|^{\sigma-1}$. Then the functional $J_\lambda\in C^1(E,\mathbb{R})$, and

$$\langle J'_\lambda(u),u\rangle=\|u\|^p-\int_\Omega F(x,u)dx-\lambda\int_\Omega a(x)\mid u\mid^q dx. \tag{3.7}$$

Proof. We define three functionals I_1, I_2 and I_3 as follows.

$$I_1(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx,\ I_2(u)=\frac{1}{\sigma}\int_\Omega F(x,u)dx \text{ and } I_3(u)=\frac{\lambda}{q}\int_\Omega a(x)|u|^q dx.$$

Claim 1: $I_1(u)\in C^1(E,\mathbb{R})$ and for any $u,v\in E$, $\langle I'_1(u),v\rangle=\int_\Omega|\nabla u|^{p-2}\nabla v.\nabla u$. For a fixed $x\in\Omega$ let us consider $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}$ defined by $\phi(\xi)=\dfrac{1}{p}|\xi|^p$. Obviously $\phi\in C^1(\mathbb{R}^n,\mathbb{R})$ and $\nabla\phi(\xi)=|\xi|^{p-2}\xi$. Thus, for all $\xi,\theta\in\mathbb{R}^n$ we have

$$\lim_{t\to 0}\frac{\phi(\xi+t\theta)-\phi(\xi)}{t}=|\xi|^{p-2}\xi.\theta.$$

As a cosequence, for $u,v\in E$ we have

$$\lim_{t\to 0}\frac{\frac{1}{p}|\nabla u+t\nabla v|^p-\frac{1}{p}|\nabla u|^p}{t}=|\nabla u|^{p-2}\nabla u.\nabla v \tag{3.8}$$

By the mean value theorem, there exists $k\in\mathbb{R}$ with $0<|k|<|t|$ such that

for each $t\in\mathbb{R}$ with $0<|t|<1$,

$$\left|\frac{\frac{1}{p}|\nabla u+t\nabla v|^p-\frac{1}{p}|\nabla u|^p}{t}\right|=\big|\ |\nabla u+kt\nabla v|^{p-2}(\nabla u+kt\nabla v)\nabla v\ \big|$$

$$\leq (|\nabla u| + |\nabla v|)^{p-1} \, |\nabla v|. \tag{3.9}$$

By Holder Inequality we have

$$\begin{aligned}
\int_\Omega \left|(|\nabla u| + |\nabla v|)^{p-1} \, |\nabla v| dx\right| &\leq \left\| \, (|\nabla u| + |\nabla v|)^{p-1} \, \right\|_{L^{p'}} \, \| \, |\nabla v| \, \|_{L^p} \\
&\leq \|v\| \left(\int_\Omega (|\nabla u| + |\nabla v|)^p dx \right)^{\frac{1}{p'}} \\
&\leq \|v\| \, 2^{\frac{p-1}{p'}} \left(\int_\Omega |\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{1}{p'}}
\end{aligned}$$

where $p + p^{'} = pp^{'}$. Hence $(|\nabla u| + |\nabla v|)^{p-1} \, |\nabla v| \in L^1(\Omega)$ due to $u, v \in E$, combining this with (3.8) and (3.9) and applying the Dominated Convergence Theorem, we obtain

$$\lim_{t \to 0} \int_\Omega \frac{\frac{1}{p}|\nabla u + t\nabla v|^p - \frac{1}{p}|\nabla u|^p}{t} dx = \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dx.$$

It means that I_1 is Gateaux differentiable and for $u \in E$,

$$\langle I_1^{'}(u), u \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla u dx = \int_\Omega |\nabla u|^p dx = \|u\|^p.$$

Next, we prove that $I_1^{'} : E \longrightarrow E^*$ is continuous. To get this aim we take a sequence $u_n \in E$ such that $u_n \longrightarrow u$ in E as $n \longrightarrow \infty$. We have

$$\lim_{n \to \infty} \int_\Omega |\nabla u_n - \nabla u|^p dx = 0.$$

Thus, up to a subsequence we have

$$\nabla u_n \longrightarrow \nabla u \text{ a.e in } \Omega \text{ as } n \longrightarrow \infty \tag{3.10}$$

and for some $h \in L^1(\Omega)$.

$$|\nabla u_n - \nabla u|^p \leq h(x) \text{ a.e } x \in \Omega. \tag{3.11}$$

Since

$$\begin{aligned}
|\nabla u_n|^p &\leq (|\nabla u| + |\nabla u_n - \nabla u|)^p \\
&\leq 2^{p-1}(|\nabla u|^p + |\nabla u_n - \nabla u|^p).
\end{aligned}$$

It follows from (3.11) that

$$|\nabla u_n|^p \leq 2^{p-1}(|\nabla u|^p + h(x)). \tag{3.12}$$

For any $u \in E$ with $\|u\| \leq 1$ and by Holder Inequality we have

$$\begin{aligned} |\langle I_1^{'}(u_n) - I_1^{'}(u), u\rangle| &= \left|\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla u dx\right| \\ &\leq \Big\| \, | \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u| \, \Big\|_{L^{p'}} \| \, |\nabla u| \, \|_{L^p} \\ &\leq \Big\| \, | \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u| \, \Big\|_{L^{p'}} . \end{aligned}$$

Hence

$$\left\|I_1^{'}(u_n) - I_1^{'}(u)\right\|_{E^*} \leq \| \, | \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \, | \, \|_{L^{p'}} . \tag{3.13}$$

First, we observe that
$\int_\Omega | \, | \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \, | \, |^{p^{'}} dx = \int_\Omega | \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \, |^{p^{'}} dx.$
It follows from (3.10) that

$$\big| \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \, \big|^{p^{'}} \longrightarrow 0 \text{ a.e } x \in \Omega$$

and from (3.12) that

$$\begin{aligned} \big| \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \, \big|^{p'} &\leq 2^{p'-1}(|\nabla u_n|^p + |\nabla u|^p) \\ &\leq 2^{p^{'}+p-1}(|\nabla u|^p + h(x)). \end{aligned}$$

Noting that $2^{p^{'}+p-1}(|\nabla u|^p + h(x)) \in L^1(\Omega)$ and applying the Dominated Convergence Theorem we have

$$\int_\Omega | \, | \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \, | \, |^{p'} dx \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Therefore

$$\| \, | \, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \, | \, \|_{L^{p'}} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.14}$$

Combining this and (3.13) we have

$$\|I_1^{'}(u_n) - I_1^{'}(u)\|_{E^*} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.15}$$

Thus $I_1^{'} : E \longrightarrow E^*$ is continuous and $I_1 \in C^1(E, \mathbb{R})$.

Claim 2: $I_2 \in C^1(E, \mathbb{R})$ and for any $u \in E$, $\langle I_2^{'}(u), u\rangle = \int_\Omega F(x, u)dx$, where $F(x,u) = \displaystyle\int_0^u f(x,s)ds$. Similar to proof I_1, let $u,h \in E$. Given $x \in \Omega$ and $0 < |t| < 1$, by the mean value theorem, there exist $\lambda \in (0,1)$ such that $0 < |\lambda| < |t| < 1$,

$$\begin{aligned}\left|\frac{\frac{1}{\sigma}F(x,u+th) - \frac{1}{\sigma}F(x,u)}{t}\right| &= \left|\frac{1}{\sigma}\frac{\partial F(x,u+\lambda h)}{\partial u}h(x)\right| \\ &\le \frac{1}{\sigma}\left|\frac{\partial F(x,u+\lambda h)}{\partial u}\right||h(x)|.\end{aligned}$$

By Remark (3.1.2) we obtain $\dfrac{|\frac{1}{\sigma}F(x,u+th) - \frac{1}{\sigma}F(x,u)|}{|t|} \le \dfrac{1}{\sigma}K(|u|+|h|)^{\sigma-1}|h|$.

Also by Holder Inequality we have

$$\begin{aligned}\int_\Omega |(|u|+|h|)^{\sigma-1}|h(x)|dx| &\le \|(|u|+|h|)^{\sigma-1}\|_{L^\gamma}\| \, |h| \, \|_{L^\sigma} \\ &\le 2^{\frac{\sigma-1}{\gamma}}\left(\int_\Omega(|u|^\sigma+|h|^\sigma)dx\right)^{\frac{1}{\gamma}}\|h\|\end{aligned}$$

where $\gamma = \dfrac{\sigma}{\sigma-1}$. Hence $\dfrac{1}{\sigma}K(|u|+|h|)^{\sigma-1}|h(x)| \in L^1$ since $u,h \in E$. It follows from the Dominated Convergence Theorem that

$$\lim_{t\to 0}\int_\Omega \frac{\frac{1}{\sigma}F(x,u+th) - \frac{1}{\sigma}F(x,u)dx}{t} = \int_\Omega \frac{1}{\sigma}\frac{\partial F(x,u)}{\partial u}hdx.$$

Thus $\langle I_2'(u),u\rangle = \displaystyle\int_\Omega \frac{1}{\sigma}\frac{\partial F(x,u)}{\partial u}udx$. By applying Property (1) we have

$$\langle I_2'(u),u\rangle = \int_\Omega F(x,u)dx. \tag{3.16}$$

To prove the Continuity of the Gateax derivative, we assume that $u_n \longrightarrow u$ in E.

By Sobolev Embedding Theorem, $u_n \longrightarrow u$ in L^p. It follows from Lemma (3.1.4) that $f(x,u_n) \longrightarrow f(x,u)$ in L^r where $r = \dfrac{p}{p-1}$. For any $u \in E$ with $\|u\| \le 1$ and by the Holder Inequality we get

$$\begin{aligned}|\langle I_2'(u_n) - I_2'(u),h\rangle| &= \left|\frac{1}{\sigma}\int_\Omega(f(x,u_n)-f(x,u))hdx\right| \\ &\le \frac{1}{\sigma}\int_\Omega|f(x,u_n)-f(x,u)|\,|h|dx \\ &\le \frac{1}{\sigma}\|f(x,u_n)-f(x,u)\|_{L^r}\|h\|_{L^p} \\ &\le \frac{1}{\sigma}\|f(x,u_n)-f(x,u)\|_{L^r}\end{aligned}$$

and so

$$\|I_2^{'}(u_n) - I_2^{'}(u)\|_{E^*} \leq \|f(x,u_n) - f(x,u)\|_L^r \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.17}$$

Thus $I_2^{'} : E \longrightarrow E^*$ is continuous and $I_2 \in C^1(E, \mathbb{R})$.

Claim 3 : $I_3 \in C^1(E, \mathbb{R})$ and for any $u \in E$, $\langle I_3^{'}(u), u\rangle = \lambda \int_\Omega a(x)|u|^q dx$. Again let $u, h \in E$ then $u, h \in L^p$. Given $x \in \Omega$ and $0 < |t| < 1$ by the mean value theorem, there exists $c \in (0,1)$ such that $0 < |c| < |t| < 1$,

$$\begin{aligned}
\left| \frac{\frac{\lambda}{q}a(x)|u+th|^q - \frac{\lambda}{q}a(x)|u|^q}{t} \right| &= \lambda\, a(x)\ \ |u+cth|^{q-1}|h(x)| \\
&\leq \lambda a(x)(|u| + |c||t||h|)^{q-1}|h(x)| \\
&\leq \lambda \|a(x)\|_\infty \,(|u| + |h|)^{q-1}\, |h(x)|. \\
&= \lambda\,(|u| + |h|)^{q-1}\, |h(x)|.
\end{aligned}$$

The Holder Inequality implies that $\lambda(|u| + |h|)^{q-1}|h(x)| \in L^1(\Omega)$. It follows from the Dominated Convergence Theorem that

$$\langle I_3'(u), h\rangle = \lambda \int_\Omega a(x)|u|^{q-2}uh dx.$$

Thus

$$\langle I_3'(u), u\rangle = \lambda \int_\Omega a(x)|u|^q dx. \tag{3.18}$$

Now we want to prove that I_3' is continuous on E. To this end let us define $f(.,u) = |u|^{q-2}u$. Assume that $u_n \longrightarrow u$ in L^q.

By Lemma (3.1.4), $f(.,u_n) \longrightarrow f(.,u)$ in L^r when $r = \frac{q}{q-1}$. For any $h \in E$ with $\|h\| \leq 1$ and by Holder Inequality we obtain

$$\begin{aligned}
|\langle I_3^{'}(u_n) - I_3^{'}(u), h\rangle| &= \left|\lambda \int_\Omega a(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)hdx\right| \\
&\leq |\lambda| \|a(x)\|_\infty \int_\Omega \big|\ |u_n|^{q-2}u_n - |u|^{q-2}u\ \big|\ |h|dx \\
&\leq |\lambda|\ \big\|\ |u_n|^{q-2}u_n - |u|^{q-2}u\ \big\|_{L^r}\ \|h\|_{L^q} \\
&\leq |\lambda|\ \big\|\ |u_n|^{q-2}u_n - |u|^{q-2}u\ \big\|_{L^r}.
\end{aligned}$$

Hence $\|I_3'(u_n) - I_3'(u)\|_{E^*} \leq |\lambda|\, \|f(u_n) - f(u)\|_{L^r} \longrightarrow 0$, so $\|I_3'(u_n) - I_3'(u)\|_{E^*} \longrightarrow 0$. Thus $I_3' : E \longrightarrow E^*$ is continous and $I_3 \in C^1(E, \mathbb{R})$.

So the functional $J_\lambda(u) = I_1(u) - I_2(u) - I_3(u)$ belongs $C^1(E,\mathbb{R})$. Further we have

$$\begin{aligned}
\langle J_\lambda'(u), u\rangle &= \langle I_1'(u), u\rangle - \langle I_2'(u), u\rangle - \langle I_3'(u), u\rangle \\
&= \|u\|^p - \int_\Omega F(x,u)dx - \lambda \int_\Omega a(x)|u|^q dx.
\end{aligned}$$

Next we prove that the second Gateaux derivative is given by

$$\langle J_\lambda^{''}(u)u, u\rangle = p\int_\Omega |\nabla u|^p dx - \sigma \int_\Omega F(x,u)dx - \lambda q \int_\Omega a(x)|u|^q dx.$$

To obtain this formula we define the functionals $\psi_1(u) = \displaystyle\int_\Omega |\nabla u|^p dx,\ \psi_2(u) = \int_\Omega F(x,u)dx$ and $\psi_3(u) = \lambda \displaystyle\int_\Omega a(x)|u|^q dx$. Now

$$\begin{aligned}
\langle \psi_1^{'}(u), h\rangle &= \frac{d}{d\epsilon}\psi_1(u+\epsilon h)|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\int_\Omega |\nabla u + \epsilon \nabla h|^p dx|_{\epsilon=0}, \\
&= \int_\Omega p|\nabla u|^{p-2}\nabla u.\nabla h dx.
\end{aligned}$$

Thus $\langle \psi_1^{'}(u), u\rangle = \int_\Omega p|\nabla u|^p$. Next

$$\begin{aligned}
\langle \psi_2^{'}(u), h\rangle &= \frac{d}{d\epsilon}\psi_2(u+\epsilon h)|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\int_\Omega F(x, u+\epsilon h)dx \\
&= \int_\Omega \frac{\partial F}{\partial u} h dx.
\end{aligned}$$

So $\langle \psi_2^{'}(u), u\rangle = \int_\Omega \frac{\partial F}{\partial u} u dx = \sigma \int_\Omega F(x,u)$ by Property (1). Finally

$$\begin{aligned}
\langle \psi_3^{'}(u), h\rangle &= \frac{d}{d\epsilon}\psi_3(u+\epsilon h)|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\lambda \int_\Omega a(x)|u+\epsilon h|^q dx|_{\epsilon=0}, \\
&= \lambda q \int_\Omega a(x)|u|^{q-2}u.h dx.
\end{aligned}$$

Thus $\langle \psi_3^{'}(u), u\rangle = \lambda q \int_\Omega a(x)|u|^q dx$. Hence

$$\begin{aligned}
\langle J_\lambda^{''}(u)u, u\rangle &= \langle \psi_1^{'}(u), u\rangle - \langle \psi_2^{'}(u), u\rangle - \langle \psi_3^{'}(u), u\rangle \\
&= p\|u\|^p - \sigma \int_\Omega F(x,u)dx - \lambda q \int_\Omega a(x)|u|^q dx \\
&= p\left(\int_\Omega F(x,u)dx + \lambda \int_\Omega a(x)|u|^q dx\right) - \sigma \int_\Omega F(x,u)dx - \lambda q \int_\Omega a(x)|u|^q dx.
\end{aligned}$$

Therfore

$$\langle J_\lambda^{''}(u)u, u\rangle = \lambda(p-q)\int_\Omega a(x)|u|^q dx - (\sigma - p)\int_\Omega F(x,u)dx \tag{3.19}$$

also

$$\begin{aligned}
\langle J_\lambda^{''}(u)u, u\rangle &= p\|u\|^p - \sigma \int_\Omega F(x,u)dx - \lambda q \int_\Omega a(x)|u|^q dx \\
&= p\|u\|^p - \sigma\left(\|u\|^p - \lambda \int_\Omega a(x)|u|^q dx\right) - \lambda q \int_\Omega a(x)|u|^q dx.
\end{aligned}$$

Thus

$$\langle J_\lambda^{''}(u)u, u\rangle = \lambda(\sigma - q)\int_\Omega a(x)|u|^q dx - (\sigma - p)\|u\|^p \tag{3.20}$$

$$\begin{aligned}
\langle J_\lambda^{''}(u)u, u\rangle &= p\|u\|^p - \sigma \int_\Omega F(x,u)dx - \lambda q \int_\Omega a(x)|u|^q dx \\
&= p\|u\|^p - \sigma \int_\Omega F(x,u)dx - q\left(\|u\|^p - \int_\Omega F(x,u)dx\right).
\end{aligned}$$

Hence

$$\langle J_\lambda^{''}(u)u, u\rangle = (p-q)\|u\|^p - (\sigma - q)\int_\Omega F(x,u)dx \tag{3.21}$$

Definition 3.1.7: (Nehari Manifold)

Assume that $J_\lambda(u) \in C^1(E, \mathbb{R})$ such that $J_\lambda'(0) = 0$. A necessary condition for $u \in E$ to be a critical point of $J_\lambda(u)$ is that $\langle J_\lambda'(u), u\rangle = 0$. This condition defines the Nehari mainfold

$$N_\lambda = \{u \in E : \langle J_\lambda'(u), u\rangle = 0, u \neq 0\}$$

where $\langle , \rangle$ denote the usual duality between E and E^*. A critical point $u \neq 0$ of J_λ is a ground state or a least energy critical point if $J_\lambda(u) = \inf\limits_N J_\lambda$. As $J_\lambda(u)$ is not bounded below on $E = W_0^{1,p}$, it is useful to consider the functional on the Nehari manifold. Thus $u \in N_\lambda$ if and only if

$$\|u\|^p - \int_\Omega F(x,u)dx - \lambda \int_\Omega a(x) \mid u \mid^q dx = 0 \tag{3.22}$$

Note that N_λ contains every nonzero solution of problem (3.1). Thus it is natural to split N_λ into three parts corresponding to local minima, local maxima and points of inflection. For this, we set

$$\begin{aligned} N_\lambda^+ &= \{u \in N_\lambda : \langle \phi_\lambda'(u), u\rangle > 0\}, \\ N_\lambda^0 &= \{u \in N_\lambda : \langle \phi_\lambda'(u), u\rangle = 0\}, \\ N_\lambda^- &= \{u \in N_\lambda : \langle \phi_\lambda'(u), u\rangle < 0\}, \end{aligned}$$

where $\phi_\lambda(u) = \langle J_\lambda'(u), u\rangle$. To state our main result, we now present some important properties of N_λ^+, N_λ^0 and N_λ^-. The following lemma shows that the minimizers on N_λ are usually critical points for J_λ.

Lemma 3.1.8:

Assume that u_0 is a local minimizer for $J_\lambda(u)$ on N_λ and that u_0 is not belonging to N_λ^0, then $J_\lambda^{'}(u_0) = 0$ in E^* (the dual space of the Sobolev space E).

Proof. Suppose that u_0 is a local minimum for $J_\lambda(u)$ on N_λ, then u_0 is a solution of the optimization problem

$$\text{minimize } J_\lambda(u) \ \text{ subject to } \langle J_\lambda^{'}(u), u\rangle = 0.$$

Hence by Lagrange multiplier there exist $\mu \in \mathbb{R}$ such that $J_\lambda^{'}(u_0) = \mu\ \phi_\lambda^{'}(u_0)$ in E^*. Thus,

$$\langle J_\lambda^{'}(u_0), u_0\rangle = \ \mu\langle \phi_\lambda^{'}(u_0), u_0\rangle.$$

Since $u_0 \in N_\lambda$ we have $0 = \langle J_\lambda^{'}(u_0), u_0\rangle = \mu\langle \phi^{'}(u_0), u_0\rangle$. But u_0 does not belong to N_λ^0, then $\langle \phi^{'}(u_0), u_0\rangle \neq 0$ therefore $\mu = 0$ and $\langle J_\lambda^{'}(u_0), u_0\rangle = 0$. Hence we get $J_\lambda^{'}(u_0) = 0$. Thus u_0 is a critical point of J_λ.

Lemma 3.1.9:

One has the following :

(i) if $u \in N_\lambda^+$, then $\lambda \displaystyle\int_\Omega a(x) \mid u \mid^q dx > 0$;

(ii) if $u \in N_\lambda^-$, then $\displaystyle\int_\Omega F(x,u)dx > 0$,

(iii) if $u \in N_\lambda^0$, then $\lambda \displaystyle\int_\Omega a(x) \mid u \mid^q dx > 0$ and $\displaystyle\int_\Omega F(x,u)dx > 0$.

Proof.

(i) $u \in N_\lambda$ iff $\|u\|^p - \displaystyle\int_\Omega F(x,u)dx - \lambda \int_\Omega a(x) \mid u \mid^q dx = 0$. Since $u \in N_\lambda^+$, then $\langle \phi^{'}(u), u\rangle > 0$. Now we consider the following two cases :

Case (1): If $\displaystyle\int_\Omega F(x,u)dx < 0$, we have

$$\lambda \int_\Omega a(x) \mid u \mid^q dx = \|u\|^p - \int_\Omega F(x,u)dx > 0.$$

Thus $\lambda \displaystyle\int_\Omega a(x) \mid u \mid^q dx > 0$.

Case (2): If $\displaystyle\int_\Omega F(x,u)dx > 0$. Since $u \in N_\lambda^+$ we have

$$\lambda(p-q)\int_\Omega a(x) \mid u \mid^q dx - (\sigma - p)\int_\Omega F(x,u)dx > 0,$$

$$\lambda(p-q)\int_\Omega a(x) \mid u \mid^q dx > (\sigma - p)\int_\Omega F(x,u)dx, \text{ or}$$

$$\lambda\int_\Omega a(x) \mid u \mid^q dx > \frac{\sigma - p}{p-q}\int_\Omega F(x,u)dx > 0.$$

Thus $\lambda\int_\Omega a(x) \mid u \mid^q dx > 0$.

(ii) $u \in N_\lambda$ iff $\|u\|^p - \int_\Omega F(x,u)dx - \lambda\int_\Omega a(x) \mid u \mid^q dx = 0$. If $u \in N_\lambda^-$, then $\langle \phi'(u), u\rangle < 0$. Now we consider the following two cases :

Case (1): If $\lambda\int_\Omega a(x) \mid u \mid^q dx = 0$. Since $u \in N_\lambda$ we have

$$\|u\|^p = \int_\Omega F(x,u)dx, \text{ but } \|u\|^p > 0.$$

Hence $\int_\Omega F(x,u)dx > 0$.

Case (2): If $\lambda\int_\Omega a(x) \mid u \mid^q dx \neq 0$. Since $u \in N_\lambda^-$ by (3.21), we have

$$(p-q)\|u\|^p - (\sigma - q)\int_\Omega F(x,u)dx < 0, \text{ or}$$

$$\int_\Omega F(x,u)dx > \frac{p-q}{\sigma - q}\|u\|^p > 0,$$

which implies $\int_\Omega F(x,u)dx > 0$.

(iii) $u \in N_\lambda$ iff $\|u\|^p - \int_\Omega F(x,u)dx - \lambda\int_\Omega a(x) \mid u \mid^q dx = 0$. Since $u \in N_\lambda^0$, then $\langle \phi'(u), u\rangle = 0$. Now by (3.20), we have

$$\lambda(\sigma - q)\int_\Omega a(x) \mid u \mid^q dx = (\sigma - p)\|u\|^p, \text{ or}$$

$$\lambda\int_\Omega a(x) \mid u \mid^q dx = \frac{\sigma - p}{\sigma - q}\|u\|^p > 0.$$

Thus $\lambda\int_\Omega a(x) \mid u \mid^q dx > 0$, and by (3.21), we get
$(p-q)\|u\|^p - (\sigma - q)\int_\Omega F(x,u)dx = 0$, $\int_\Omega F(x,u)dx = \frac{p-q}{\sigma-q}\|u\|^p > 0$.
Therefore $\int_\Omega F(x,u)dx > 0$.

Lemma 3.1.10:

If $0 < |\lambda| < \lambda_0$, where $\lambda_0 = \frac{q}{p}\left(\frac{\sigma - p}{\sigma - q}\right) S_q^{\frac{q}{p}} \left(\frac{p-q}{(\sigma-q)K} S_\sigma^{\frac{\sigma}{p}}\right)^{\dfrac{p-q}{\sigma-p}}$ then $N_\lambda^0 = \phi$

Proof. Suppose otherwise that $0 <| \lambda |< \lambda_0$ such that $N_\lambda^0 \neq \phi$. Then for $u \in N_\lambda^0$, we have

$$0 = \langle \phi_\lambda^{'}(u), u\rangle = \lambda(\sigma - q)\int_\Omega a(x)|u|^q dx - (\sigma - p)\|u\|^p \tag{3.23}$$

$$= (p-q)\|u\|^p - (\sigma - q)\int_\Omega F(x,u)dx. \tag{3.24}$$

Using Property (2) and by Remark (3.1.3), we obtain

$\int_\Omega F(x,u)dx \leq \left|\int_\Omega F(x,u)dx\right| \leq \int_\Omega |F(x,u)|dx \leq K\int_\Omega |u|^\sigma dx \leq KS_\sigma^{\frac{-\sigma}{p}}\|u\|^\sigma.$

Hence, it follows from (3.24) that

$$\|u\|^p = \frac{\sigma - q}{p-q}\int_\Omega F(x,u)dx \leq \frac{\sigma-q}{p-q} KS_\sigma^{\frac{-\sigma}{p}}\|u\|^\sigma,$$

then

$$\|u\| \geq \left(\frac{(p-q)S_\sigma^{\frac{\sigma}{p}}}{(\sigma-q)K}\right)^{\frac{1}{\sigma-p}}. \tag{3.25}$$

On the other hand, from Holder Inequality, Condition (1), Equation (3.23) and by Remark (3.1.3) we have

$$\|u\|^p = \frac{\lambda(\sigma-q)}{\sigma-p}\int_\Omega a(x)|u|^q dx \leq |\lambda|\ \frac{\sigma-q}{\sigma-p}\ \|a\|_\infty \int_\Omega |u|^q dx,$$

$$\leq |\lambda|\ \frac{\sigma-q}{\sigma-p}\ S_q^{\frac{-q}{p}}\|u\|^q.$$

So

$$\|u\| \leq \left(|\lambda|\ \frac{\sigma-q}{\sigma-p}\ S_q^{\frac{-q}{p}}\right)^{\frac{1}{p-q}} \tag{3.26}$$

Combining (3.25) and (3.26), we have $|\lambda| \geq \lambda_0$ a contradiction. Therefore $N_\lambda^0 = \phi$ for $0 < |\lambda| < \lambda_0$.

We remark that by Lemma (3.1.10), for $0 < |\lambda| < \lambda_0$, $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ and define

$$\theta_\lambda = \inf_{u\in N_\lambda} J_\lambda(u),\ \ \theta_\lambda^+ = \inf_{u\in N_\lambda^+} J_\lambda(u),\ \ \theta_\lambda^- = \inf_{u\in N_\lambda^-} J_\lambda(u). \tag{3.27}$$

Then, we have the following.

Lemma 3.1.11:

If $0 < |\lambda| < \lambda_0$, then $\theta_\lambda \leq \theta_\lambda^+ < 0, \theta_\lambda^- > d_0$ for some $d_0 > 0$ depending on $p, q, \sigma, k, \lambda, S_q$ and S_σ.

Proof. Let $u \in N_\lambda^+$. Then from (3.21) we have

$$(p-q)\|u\|^p - (\sigma - q)\int_\Omega F(x,u)dx > 0. \text{ Thus}$$

$$\frac{p-q}{\sigma-q}\|u\|^p > \int_\Omega F(x,u)dx. \tag{3.28}$$

So

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\|u\|^p - \frac{1}{\sigma}\int_\Omega F(x,u)dx - \frac{\lambda}{q}\int_\Omega a(x)|u|^q dx \\ &= \frac{1}{p}\|u\|^p - \frac{1}{\sigma}\int_\Omega F(x,u)dx - \frac{1}{q}\left(\|u\|^p - \int_\Omega F(x,u)dx\right) \\ &= \frac{q-p}{pq}\|u\|^p + \frac{\sigma-q}{\sigma q}\int_\Omega F(x,u)dx. \end{aligned}$$

By (3.28) we have

$$\begin{aligned} J_\lambda(u) &< \frac{q-p}{pq}\,\|u\|^p + \frac{p-q}{\sigma q}\,\|u\|^p, \\ &< \left(\frac{q-p}{pq} + \frac{p-q}{\sigma q}\right)\|u\|^p, \\ &< \frac{-(p-q)(\sigma-p)}{\sigma p q}\|u\|^p < 0. \end{aligned}$$

Hence $J_\lambda(u) < 0$, since $N_\lambda^+ \subset N_\lambda$, it follows that $\inf\limits_{u \in N_\lambda} J_\lambda(u) \leq \inf\limits_{u \in N_\lambda^+} J_\lambda(u)$, so by the definition of θ_λ and θ_λ^+ we obtain $\theta_\lambda \leq \theta_\lambda^+ < 0$.

Now, let $u \in N_\lambda^-$, then from (3.21) we have

$$(p-q)\|u\|^p - (\sigma-q)\int_\Omega F(x,u)dx < 0.$$

Using Property (2) and by Remark (3.1.3) we have

$\dfrac{p-q}{\sigma-q}\|u\|^p < \left|\int_\Omega F(x,u)dx\right| < \int_\Omega |F(x,u)|dx < K\int_\Omega |u|^\sigma dx \leq KS_\sigma^{\frac{-\sigma}{p}}\|u\|^\sigma$. Therefore

$\|u\|^p < \dfrac{\sigma-q}{p-q}KS_\sigma^{\frac{-\sigma}{p}}\|u\|^\sigma$, or we write

$$\|u\| > \left(\frac{p-q}{(\sigma-q)K}S_\sigma^{\frac{\sigma}{p}}\right)^{\frac{1}{\sigma-p}} \forall u \in N_\lambda^-. \tag{3.29}$$

Thus

$J_\lambda(u) \geq \dfrac{\sigma-p}{\sigma p}\|u\|^p - |\lambda|S_q^{\frac{-q}{p}}\dfrac{\sigma-q}{\sigma q}\|u\|^q = \|u\|^q\left(\dfrac{\sigma-p}{\sigma p}\|u\|^{p-q} - |\lambda|S_q^{\frac{-q}{p}}\dfrac{\sigma-q}{\sigma q}\right)$. Hence

$$J_\lambda(u) > \left(\frac{p-q}{(\sigma-q)K}S_\sigma^{\frac{\sigma}{p}}\right)^{\frac{q}{\sigma-p}}\left(\frac{\sigma-p}{\sigma p}\left(\frac{p-q}{(\sigma-q)K}S_\sigma^{\frac{\sigma}{p}}\right)^{\frac{p-q}{\sigma-p}} - |\lambda|S_q^{\frac{-q}{p}}\frac{\sigma-q}{\sigma q}\right).$$

Therefore $J_\lambda(u) > d_0$ for some $d_0 > 0$, where

$$d_0 = \left(\frac{p-q}{(\sigma-q)K}S_\sigma^{\frac{\sigma}{p}}\right)^{\frac{q}{\sigma-p}}\left(\frac{\sigma-p}{\sigma p}\left(\frac{p-q}{(\sigma-q)K}S_\sigma^{\frac{\sigma}{p}}\right)^{\frac{p-q}{\sigma-p}} - |\lambda|S_q^{\frac{-q}{p}}\frac{\sigma-q}{\sigma q}\right).$$

In order to prove that the functional J_λ has a minimum, we would need to know that J_λ is bounded below. Of course, this is necessary but not enough to guarantee the existence of a minimizer for J_λ. In the next lemma we prove that J_λ is bounded below and grows rapidly "coercive" at the "extremes" of N_λ.

Lemma 3.1.12:

The energy functional J_λ is coercive and bounded below on N_λ

Proof. If $u \in N_\lambda$, then $\|u\|^p - \int_\Omega F(x,u)dx - \lambda \int_\Omega a(x)|u|^q dx = 0$

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\|u\|^p - \frac{1}{\sigma}\int_\Omega F(x,u)dx - \frac{\lambda}{q}\int_\Omega a(x)|u|^q dx \\
J_\lambda(u) &= \frac{1}{p}\|u\|^p - \frac{1}{\sigma}\left(\|u\|^p - \lambda\int_\Omega a(x)|u|^q dx\right) - \frac{\lambda}{q}\int_\Omega a(x)|u|^q dx \\
&= \frac{\sigma - p}{\sigma p}\|u\|^p - \lambda\frac{\sigma - q}{q\sigma}\int_\Omega a(x) \mid u \mid^q dx
\end{aligned}$$

by Remark (3.1.3) and Condition (1) we obtain

$$\begin{aligned}
J_\lambda(u) &\geq \frac{\sigma - p}{\sigma p}\|u\|^p - (\frac{\sigma - q}{q\sigma})\|\lambda a(x)\|_\infty \int_\Omega \mid u \mid^q dx \\
&\geq \frac{\sigma - p}{\sigma p}\|u\|^p - (\frac{\sigma - q}{q\sigma}) \mid \lambda \mid \int_\Omega \mid u \mid^q dx \\
&\geq \frac{\sigma - p}{\sigma p}\|u\|^p - (\frac{\sigma - q}{q\sigma})(\frac{\mid \lambda \mid}{s_q^{\frac{q}{p}}})\|u\|^q.
\end{aligned}$$

Since $1 < q < p,\ J_\lambda(u) \longrightarrow \infty$ as $\|u\| \longrightarrow \infty$. Therefore $J_\lambda(u)$ is coercive and bounded below on N_λ.

Definition 3.1.13:

For $u \in E$ with $\int_\Omega F(x,u) > 0$ define T to be

$T = \left(\frac{(p-q)\|u\|^p}{(\sigma - q)\int_\Omega F(x,u)dx}\right)^{\frac{1}{\sigma - p}} > 0$. Then the following result holds.

Lemma 3.1.14:

For each $u \in E$ with $\int_\Omega F(x,u)dx > 0$ one has the following:

(i) if $\lambda \int_\Omega a(x)|u|^q dx \leq 0$, then there exists a unique $t^- > T$ such that $t^- u \in N_\lambda^-$ and $J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu)$

(ii) if $\lambda \int_\Omega a(x)|u|^q dx > 0$, then there are unique $0 < t^+ < T < t^-$ such that $(t^- u, t^+ u) \in N_\lambda^- \times N_\lambda^+$ and $J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu)$; $J_\lambda(t^+ u) = \inf_{0 \leq t \leq T} J_\lambda(tu)$.

Proof. We fix $u \in E$ with $\displaystyle\int_\Omega F(x,u)dx > 0$ and we define the maps

$m_u(t) : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ by

$$m_u(t) = t^{p-q}\|u\|^p - t^{\sigma-q}\int_\Omega F(x,u)dx \text{ for } t \geq 0. \tag{3.30}$$

$$\begin{aligned}
\langle J'_\lambda(tu),(tu)\rangle &= \|tu\|^p - \int_\Omega F(x,tu)dx - \lambda\int_\Omega a(x)|tu|^q dx \\
&= (t)^p\|u\|^p - (t)^\sigma\int_\Omega F(x,u)dx - \lambda(t)^q\int_\Omega a(x)|u|^q dx \\
&= (t)^q\left((t)^{p-q}\|u\|^p - (t)^{\sigma-q}\int_\Omega F(x,u)dx - \lambda\int_\Omega a(x)|u|^q dx\right). \\
&= (t)^q\left(m_u(t) - \lambda\int_\Omega a(x)|u|^q dx\right) = 0 \text{ iff } m_u(t) = \lambda\int_\Omega a(x)|u|^q dx.
\end{aligned}$$

Clearly for $t > 0, tu \in N_\lambda$ iff t is a solution of $m_u(t) = \lambda\displaystyle\int_\Omega a(x)|u|^q dx$. Now

$$m'_u(t) = (p-q)t^{p-q-1}\|u\|^p - (\sigma-q)t^{\sigma-q-1}\int_\Omega F(x,u)dx$$

and $m'_u(t) = 0$ implies that $(\sigma-q)t^{\sigma-q-1}\displaystyle\int_\Omega F(x,u)dx\left[\frac{(p-q)\|u\|^p}{(\sigma-q)t^{\sigma-p}\int_\Omega F(x,u)dx} - 1\right] = 0$

giving $t = \left(\dfrac{(p-q)\|u\|^p}{(\sigma-q)\int_\Omega F(x,u)dx}\right)^{\frac{1}{\sigma-p}}$. Hence $m'_u(t) = 0$ for

$T = \left(\dfrac{(p-q)\|u\|^p}{(\sigma-q)\int_\Omega F(x,u)dx}\right)^{\frac{1}{\sigma-p}}$, $m'_u(t) > 0$ for $t \in (0,T)$ and $m'_u(t) < 0$ for $t \in (T,\infty)$.

Then $m_u(t)$ has a maximum at $t = T$, increasing for $t \in (0,T)$ and decreasing for $t \in (T,\infty)$. Moreover $m_u(T) =$

$$= \left(\frac{(p-q)\|u\|^p}{(\sigma-q)\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}}\|u\|^p - \left(\frac{(p-q)\|u\|^p}{(\sigma-q)\int_\Omega F(x,u)dx}\right)^{\frac{\sigma-q}{\sigma-p}}\int_\Omega F(x,u)dx$$

$$= \left(\frac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}}\left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}}\|u\|^p - \left(\frac{p-q}{\sigma-q}\right)^{\frac{\sigma-q}{\sigma-p}}\left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{\sigma-q}{\sigma-p}}\int_\Omega F(x,u)dx.$$

Since $\dfrac{p-q}{\sigma-p} = \dfrac{\sigma-\sigma+p-q}{\sigma-p} = \dfrac{\sigma-q}{\sigma-p} - 1$, Then

$$m_u(T) = \left(\frac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}} \left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{\sigma-q}{\sigma-p}} \frac{\int_\Omega F(x,u)dx}{\|u\|^p}\|u\|^p -$$

$$\left(\frac{p-q}{\sigma-q}\right)^{\frac{\sigma-q}{\sigma-p}} \left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{\sigma-q}{\sigma-p}} \int_\Omega F(x,u)dx$$

$$= \left[\left(\frac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}} - \left(\frac{p-q}{\sigma-q}\right)^{\frac{\sigma-q}{\sigma-p}}\right] \left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{\sigma-q}{\sigma-p}} \int_\Omega F(x,u)dx$$

Or we write

$$m_u(T) = \|u\|^q \left[\left(\frac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}} - \left(\frac{p-q}{\sigma-q}\right)^{\frac{\sigma-q}{\sigma-p}}\right] \times \left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{\sigma-q}{\sigma-p}} \frac{\int_\Omega F(x,u)dx}{\|u\|^q}.$$

Since $\dfrac{\sigma-q}{\sigma-p} = \dfrac{p-p+\sigma-q}{\sigma-p} = \dfrac{p-q}{\sigma-p} + 1$

$$m_u(T) = \|u\|^q \left[\left(\frac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}} - \left(\frac{p-q}{\sigma-q}\right)^{\frac{\sigma-q}{\sigma-p}}\right] \times \left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}}$$

$$\times \frac{\|u\|^p}{\int_\Omega F(x,u)dx} \frac{\int_\Omega F(x,u)dx}{\|u\|^q}$$

$$= \|u\|^q \left[\left(\frac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}} - \left(\frac{p-q}{\sigma-q}\right)^{\frac{\sigma-q}{\sigma-p}}\right] \times \left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}} \frac{\|u\|^p}{\|u\|^q}$$

$$\left(\frac{\|u\|^p}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}} \|u\|^{p-q} = \frac{\left(\|u\|^{\frac{p}{\sigma-p}}\|u\|\right)^{p-q}}{\left(\int_\Omega F(x,u)dx\right)^{\frac{p-q}{\sigma-p}}} = \frac{\left(\|u\|^{\frac{p}{\sigma-p}+1}\right)^{p-q}}{\left(\int_\Omega F(x,u)dx\right)^{\frac{p-q}{\sigma-p}}} =$$

$$\frac{\left(\|u\|^{\frac{\sigma}{\sigma-p}}\right)^{p-q}}{\left(\int_\Omega F(x,u)dx\right)^{\frac{p-q}{\sigma-p}}} = \frac{(\|u\|^\sigma)^{\frac{p-q}{\sigma-p}}}{\left(\int_\Omega F(x,u)dx\right)^{\frac{p-q}{\sigma-p}}} = \left(\frac{\|u\|^\sigma}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}}$$

Hence $m_u(T) = \|u\|^q \left[\left(\dfrac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}} - \left(\dfrac{p-q}{\sigma-q}\right)^{\frac{\sigma-q}{\sigma-p}}\right] \times \left(\dfrac{\|u\|^\sigma}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}}$.

(i) Suppose that $\lambda\displaystyle\int_\Omega a(x)|u|^q dx \le 0$, then there is a unique $t^- > T$ such that $m_u(t^-) = \lambda\displaystyle\int_\Omega a(x)|u|^q dx$. Now by (3.21)

$$\begin{aligned}
\langle \phi'_\lambda(t^-u), t^-u\rangle &= (p-q)\|t^-u\|^p - (\sigma-q)\int_\Omega F(x,t^-u)dx \\
&= (p-q)(t^-)^p\|u\|^p - (\sigma-q)(t^-)^\sigma\int_\Omega F(x,u)dx \\
&= (t^-)^{1+q}\left((p-q)(t^-)^{p-q-1}\|u\|^p - (\sigma-q)(t^-)^{\sigma-q-1}\int_\Omega F(x,u)dx\right) \\
&= (t^-)^{1+q}(m'_u(t^-)) < 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle J'_\lambda(t^-u), (t^-u)\rangle &= \|t^-u\|^p - \int_\Omega F(x,t^-u)dx - \lambda\int_\Omega a(x)|t^-u|^q dx \\
&= (t^-)^p\|u\|^p - (t^-)^\sigma\int_\Omega F(x,u)dx - \lambda(t^-)^q\int_\Omega a(x)|u|^q dx \\
&= (t^-)^q\left((t^-)^{p-q}\|u\|^p - (t^-)^{\sigma-q}\int_\Omega F(x,u)dx - \lambda\int_\Omega a(x)|u|^q dx\right). \\
&= (t^-)^q\left(m_u(t^-) - \lambda\int_\Omega a(x)|u|^q dx\right) = 0.
\end{aligned}$$

Thus $t^-u \in N_\lambda^-$, since $J'_\lambda(tu) > 0$ for $0 \le t \le t^-u$ and $J'_\lambda(tu) < 0$ for $t \ge t^-u$.

Then $J_\lambda(t^-u) = \sup\limits_{t\ge 0} J_\lambda(tu)$.

To prove case (ii) we need the following

$$\begin{aligned}
m_u(T) &= \|u\|^q\left(\frac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}}\left[1-\left(\frac{p-q}{\sigma-q}\right)^{\frac{\sigma-q}{\sigma-p}-\frac{p-q}{\sigma-p}}\right]\times\left(\frac{\|u\|^\sigma}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}} \\
&= \|u\|^q\left(\frac{\sigma-p}{\sigma-q}\right)\left(\frac{p-q}{\sigma-q}\right)^{\frac{p-q}{\sigma-p}}\times\left(\frac{\|u\|^\sigma}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}}.
\end{aligned}$$

Since $\dfrac{\sigma-q}{\sigma-p} - \dfrac{p-q}{\sigma-p} = \dfrac{\sigma-p}{\sigma-p} = 1;\ 1 - \dfrac{p-q}{\sigma-q} = \dfrac{\sigma-p}{\sigma-q}$.

$\int_\Omega F(x,u)dx \le \int_\Omega |F(x,u)|dx \le K\int_\Omega |u|^\sigma dx \le KS_\sigma^{\frac{-\sigma}{p}}\|u\|^\sigma$. Thus

$$\frac{1}{\int_\Omega F(x,u)dx} \ge \frac{S_\sigma^{\frac{\sigma}{p}}}{K\|u\|^\sigma} \text{ or write } \frac{\|u\|^\sigma}{\int_\Omega F(x,u)dx} \ge \frac{S_\sigma^{\frac{\sigma}{p}}}{K},$$

$\left(\frac{\|u\|^\sigma}{\int_\Omega F(x,u)dx}\right)^{\frac{p-q}{\sigma-p}} \ge \left(\frac{S_\sigma^{\frac{\sigma}{p}}}{K}\right)^{\frac{p-q}{\sigma-p}}$. Hence $m_u(T) \ge \|u\|^q\left(\frac{\sigma-p}{\sigma-q}\right)\left(\frac{(p-q)S_\sigma^{\frac{\sigma}{p}}}{(\sigma-q)K}\right)^{\frac{p-q}{\sigma-p}}$.

(ii) Suppose that $\lambda\int_\Omega a(x)|u|^q dx > 0$, then by Condition (1), Remark (3.1.3) and the fact that $|\lambda| < \lambda_0$, we obtain
$m_u(0) = 0 < \lambda\int_\Omega a(x)|u|^q dx \le |\lambda|\|a(x)\|_\infty\int_\Omega |u|^q dx \le |\lambda|\int_\Omega |u|^q dx < \lambda_0\ S_q^{\frac{-q}{p}}\|u\|^q < m_u(T)$. Since $m_u(T) > \lambda\int_\Omega a(x)|u|^q dx$, then the equation $m_u(t) = \lambda\int_\Omega a(x)|u|^q dx$
has exactly two solution t^+ and t^- such that $0 < t^+ < T < t^-$,
$m_u(t^+) = \lambda\int_\Omega a(x)|u|^q dx = m_u(t^-)$ and $m_u'(t^-) < 0 < m_u'(t^+)$. Using similar argument to case (i) we get $(t^-u, t^+u) \in N_\lambda^- \times N_\lambda^+$, and $J_\lambda(t^+u) \le J_\lambda(tu) \le J_\lambda(t^-u)$ for all $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \le J_\lambda(tu)$ for all $t \in [0, t^+]$. Therefore $J_\lambda(t^-u) = \sup_{t\ge 0} J_\lambda(tu)$ and $J_\lambda(t^+u) = \inf_{0\le t\le T} J_\lambda(tu)$.

Definition 3.1.15:

For each $u \in E$ with $\lambda\int_\Omega a(x)|u|^q dx > 0$ define $\tilde{T} > 0$

to be $\tilde{T} = \left(\frac{(\sigma-q)\lambda\int_\Omega a(x)|u|^q dx}{(\sigma-p)\|u\|^p}\right)^{\frac{1}{p-q}}$.

Lemma 3.1.16:

For each $u \in E$ with $\lambda\int_\Omega a(x)|u|^q dx > 0$, one has the following:

(i) if $\int_\Omega F(x,u)dx \le 0$, then there exists a unique $0 < t^+ < \tilde{T}$
such that $t^+u \in N_\lambda^+$ and $J_\lambda(t^+u) = \inf_{t\ge 0} J_\lambda(tu)$;

(ii) if $\int_\Omega F(x,u)dx > 0$, then there are unique $0 < t^+ < \tilde{T} < t^-$ such that
$(t^-u, t^+u) \in N_\lambda^- \times N_\lambda^+$ and $J_\lambda(t^-u) = \sup_{t\ge 0} J_\lambda(tu)$, $J_\lambda(t^+u) = \inf_{0\le t\le \tilde{T}} J_\lambda(tu)$.

Proof. For $u \in E$ with $\lambda \displaystyle\int_\Omega a(x)|u|^q dx > 0$, we can take $\tilde{m}_u(t) : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ by $\tilde{m}_u(t) = t^{p-\sigma}\|u\|^p - \lambda t^{q-\sigma} \displaystyle\int_\Omega a(x)|u|^q dx$. Clearly for $t > 0$, $tu \in N_\lambda$ iff t is a solution of $\tilde{m}_u(t) = \int_\Omega F(x,u)dx$. Since

$$
\begin{aligned}
\langle J_\lambda'(tu), (tu)\rangle &= \|tu\|^p - \int_\Omega F(x,tu)dx - \lambda \int_\Omega a(x)|tu|^q dx \\
&= (t)^p\|u\|^p - (t)^\sigma \int_\Omega F(x,u)dx - \lambda(t)^q \int_\Omega a(x)|u|^q dx \\
&= (t)^\sigma \left((t)^{p-\sigma}\|u\|^p - \int_\Omega F(x,u)dx - \lambda(t)^{q-\sigma} \int_\Omega a(x)|u|^q dx \right). \\
&= (t)^\sigma \left(\tilde{m}(t) - \int_\Omega F(x,u)dx \right) = 0 \ \text{ iff } \tilde{m}(t) = \int_\Omega F(x,u)dx.
\end{aligned}
$$

Now $\tilde{m_u}'(t) = (p-\sigma)t^{p-\sigma-1}\|u\|^p - \lambda(q-\sigma)t^{q-\sigma-1} \displaystyle\int_\Omega a(x)|u|^q dx$ and $m_u'(t) = 0$.

This implies that $\tilde{T} = \left(\dfrac{\lambda(\sigma-q)\int_\Omega a(x)|u|^q dx}{(\sigma-p)\|u\|^p} \right)^{\frac{1}{p-q}}$.

Therefore $\tilde{m_u}'(t) > 0$ for $t \in [0, \tilde{T})$ and $\tilde{m}'(t) < 0$ for $t \in (\tilde{T}, \infty)$, then $\tilde{m}_u(t)$ a chieves its maximum at $\tilde{T}$, increasing for $t \in [0, \tilde{T})$, decreasing for $t \in (\tilde{T}, \infty)$, and $\tilde{m}_u(t) \longrightarrow -\infty$ as $t \longrightarrow 0^+$. Since

$$
\begin{aligned}
\tilde{m}_u(t) &= t^{p-\sigma}\|u\|^p - \lambda t^{q-\sigma} \int_\Omega a(x)|u|^q dx \\
&= t^{q-\sigma} \left(t^{p-q}\|u\|^p - \lambda \int_\Omega a(x)|u|^q dx \right) \\
&= -t^{q-\sigma} \left(\lambda \int_\Omega a(x)|u|^q dx - t^{p-q}\|u\|^p \right) \\
&\leq -t^{q-\sigma}\lambda \int_\Omega a(x)|u|^q dx \quad \text{for small } t \\
&= \frac{-\lambda \int_\Omega a(x)|u|^q dx}{t^{\sigma-q}} \longrightarrow -\infty \text{ as } t \longrightarrow 0^+.
\end{aligned}
$$

For $\tilde{T} > 0$, $\tilde{m}_u(\tilde{T}) =$

$$\begin{aligned}
&= \left(\frac{(\sigma-q)\lambda\int_\Omega a(x)|u|^q dx}{(\sigma-p)\|u\|^p}\right)^{\frac{p-\sigma}{p-q}} \|u\|^p - \lambda\left(\frac{(\sigma-q)\lambda\int_\Omega a(x)|u|^q dx}{(\sigma-p)\|u\|^p}\right)^{\frac{q-\sigma}{p-q}} \int_\Omega a(x)|u|^q dx \\
&= \left(\frac{(\sigma-q)\lambda\int_\Omega a(x)|u|^q dx}{(\sigma-p)\|u\|^p}\right)^{\frac{q-\sigma}{p-q}} \left[\frac{(\sigma-q)\lambda\int_\Omega a(x)|u|^q dx}{(\sigma-p)\|u\|^p}\|u\|^p - \lambda\int_\Omega a(x)|u|^q dx\right] \\
&= \left(\frac{(\sigma-q)\lambda\int_\Omega a(x)|u|^q dx}{(\sigma-p)\|u\|^p}\right)^{\frac{q-\sigma}{p-q}} \lambda\left(\frac{\sigma-q}{\sigma-p} - 1\right)\int_\Omega a(x)|u|^q dx \\
&= \frac{p-q}{\sigma-p}\lambda\int_\Omega a(x)|u|^q dx \left(\frac{(\sigma-q)\lambda\int_\Omega a(x)|u|^q dx}{(\sigma-p)\|u\|^p}\right)^{\frac{q-\sigma}{p-q}} > 0
\end{aligned}$$

(i) Suppose that $\int_\Omega F(x,u)dx \leq 0$ then there is a unique $t^+ < \tilde{T}$ such that $\tilde{m_u}(t^+) = \int_\Omega F(x,u)dx$ and $\tilde{m_u}'(t^+) > 0$. Now by (3.20)

$$\begin{aligned}
\langle \phi'_\lambda(t^+u), t^+u\rangle &= \lambda(\sigma-q)\int_\Omega a(x)|t^+u|^q - (\sigma-p)\|t^+u\|^p \\
&= \lambda(\sigma-q)(t^+)^q\int_\Omega a(x)|u|^q - (\sigma-p)(t^+)^p\|u\|^p \\
&= (t^+)^{\sigma+1}\left(-\lambda(q-\sigma)(t^+)^{q-\sigma-1}\int_\Omega a(x)|u|^q - -(p-\sigma)(t^+)^{p-\sigma-1}\|u\|^p\right) \\
&= (t^+)^{\sigma+1}\left((p-\sigma)(t^+)^{p-\sigma-1}\|u\|^p - \lambda(q-\sigma)(t^+)^{q-\sigma-1}\int_\Omega a(x)|u|^q\right) \\
&= (t^+)^{\sigma+1}\left(\tilde{m}_u'(t^+)\right) > 0, \text{ for } t^+ > 0
\end{aligned}$$

and

$$\begin{aligned}
\langle J'_\lambda(t^+u),(t^+u)\rangle &= \|t^+u\|^p - \int_\Omega F(x,t^+u)dx - \lambda\int_\Omega a(x)|t^+u|^q dx \\
&= (t^+)^p\|u\|^p - (t^+)^\sigma\int_\Omega F(x,u)dx - \lambda(t^+)^q\int_\Omega a(x)|u|^q dx \\
&= (t^+)^\sigma\left((t^+)^{p-\sigma}\|u\|^p - \int_\Omega F(x,u)dx - \lambda(t^+)^{q-\sigma}\int_\Omega a(x)|u|^q dx\right). \\
&= (t^+)^\sigma\left(\tilde{m}_u(t^+) - \int_\Omega F(x,u)dx\right) \\
&= (t^+)^\sigma\left(\tilde{m}_u(t^+) - \tilde{m}_u(t^+)\right) = 0.
\end{aligned}$$

Hence $t^+u \in N_\lambda^+$ for all $0 < t^+ < \tilde{T}$. Further since $\forall t : 0 < t < \tilde{T}$, $J'_\lambda(tu) > 0$, and $\forall t : t > \tilde{T}$, $J'_\lambda(tu) < 0$ then $J_\lambda(t^+u) = \inf\limits_{t\geq 0} J_\lambda(tu)$.

(ii) Suppose that $\int_\Omega F(x,u)dx > 0$, then by Property (2) and Remark (3.1.3), we obtain $0 < \int_\Omega F(x,u)dx \leq \left|\int_\Omega F(x,u)dx\right| \leq \int_\Omega |F(x,u)|dx < K\int_\Omega |u|^\sigma < K\ S_\sigma^{\frac{-\sigma}{p}}\|u\|^\sigma < \tilde{m}_u(\tilde{T})$. Since $\tilde{m}_u(\tilde{T}) > \int_\Omega F(x,u)dx$, then the equation $\tilde{m}_u(t) = \int_\Omega F(x,u)dx$ has exactly two solution t^+ and t^- such that $0 < t^+ < \tilde{T} < t^-$, $\tilde{m}_u(t^+) = \int_\Omega F(x,u)dx = \tilde{m}_u(t^-)$ and $\tilde{m}'_u(t^-) < 0 < \tilde{m}'_u(t^+)$. Thus we get $(t^-u, t^+u) \in N_\lambda^- \times N_\lambda^+$, and $J_\lambda(t^+u) \leq J_\lambda(tu) \leq J_\lambda(t^-u)$ for all $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \leq J_\lambda(tu)$ for all $t \in [0, t^+]$. Therefore $J_\lambda(t^-u) = \sup\limits_{t\geq 0} J_\lambda(tu)$ and $J_\lambda(t^+u) = \inf\limits_{0\leq t\leq\tilde{T}} J_\lambda(tu)$.

To prove the main result we need the follwing Theorem.

Theorem 3.1.17:

If E is a Banach space and $J_\lambda(u)$ bounded from below on N_λ then there exist a minimizing sequences u_n in N_λ such that $J_\lambda(u_n) \longrightarrow \theta_\lambda$ and $J'_\lambda(u_n) \longrightarrow 0$ in E^*. Since the functional bounded from below on N_λ^+ and N_λ^- then we have the following

(i) There exist a minimizing sequences u_n^+ in N_λ^+ such that

$$J_\lambda(u_n^+) = \theta_\lambda^+ + o(1), J_\lambda^{'}(u_n^+) = o(1) \text{ in } E^*$$

(ii) There exist a minimizing sequences u_n^- in N_λ^- such that

$$J_\lambda(u_n^-) = \theta_\lambda^- + o(1), J_\lambda^{'}(u_n^-) = o(1) \text{ in } E^*$$

Proof. For the proof see [23].

3.2 Existence of Positive Solutions

In this section we introduce a simple proof of the existence of two positive solutions of Equation (3.1), one in N_λ^+ and one in N_λ^-.

Theorem 3.2.1:

Under the assumptions (1),(2) and (3), there exists $\lambda_0 > 0$ such that for all $0 < |\lambda| < \lambda_0$, problem (3.1) has at least two nontrivial nonnegative solutions.

The proof of this theorem is a direct result from the following theorems (3.2.2) and (3.2.3)

In the next theorem we establish the existence of a local minimum for J_λ on N_λ^+.

Theorem 3.2.2:

If $0 < |\lambda| < \lambda_0$, then problem (3.1) has a positive solution u_0^+ in N_λ^+ such that

$$J_\lambda(u_0^+) = \theta_\lambda^+$$

Proof. Since J_λ is bounded below on N_λ^+, then there exist a minimizing sequence $\{u_n^+\} \subset N_\lambda^+$ such that

$$\lim_{n\longrightarrow\infty} J_\lambda(u_n^+) = \inf_{u\in N_\lambda^+} J_\lambda(u).$$

Since E is a Banach space, this sequence contains a weakly convergent subsequence u_n to u_0^+ the weak limit of u_n. By Theorem (2.2.13), we may assume that u_n converges strongly in L^q and in L^σ, $u_n \rightharpoonup u_0^+$ weakly in E, implies $u_n \longrightarrow u_0^+$ strongly in L^q and in L^σ this implies that is

$$\int_\Omega a(x)|u_n|^q dx \longrightarrow \int_\Omega a(x)|u_0^+|^q dx.$$

Next we will show that $\displaystyle\int_\Omega F(x,u_n)dx \longrightarrow \int_\Omega F(x,u_0^+)dx$ as $n \longrightarrow \infty$. By Lemma (3.1.4), we have $\dfrac{\partial F(x,u_n)}{\partial u} \in L^\gamma$, $\dfrac{\partial F(x,u_n)}{\partial u} \longrightarrow \dfrac{\partial F(x,u_0^+)}{\partial u}$ in L^γ, where $\gamma = \dfrac{\sigma}{\sigma-1}$.

On the other hand, it follows from the Holder Inequality that

$$\int_\Omega \left| u_n \frac{\partial F(x,u_n)}{\partial u} - u_0^+ \frac{\partial F(x,u_0^+)}{\partial u} dx \right| =$$

$$= \int_\Omega \left| u_n \frac{\partial F(x,u_n)}{\partial u} - u_0^+ \frac{\partial F(x,u_n)}{\partial u} + u_0^+ \frac{\partial F(x,u_n)}{\partial u} - u_0^+ \frac{\partial F(x,u_0^+)}{\partial u} \right| dx$$

$$\leq \int_\Omega \left| u_n \frac{\partial F(x,u_n)}{\partial u} - u_0^+ \frac{\partial F(x,u_n)}{\partial u} \right| dx + \int_\Omega |u_0^+| \left| \frac{\partial F(x,u_n)}{\partial u} - \frac{\partial F(x,u_0^+)}{\partial u} \right| dx$$

$$\leq \int_\Omega |u_n - u_0^+| \left| \frac{\partial F(x,u_n)}{\partial u} dx \right| + \int_\Omega |u_0^+| \left| \frac{\partial F(x,u_n)}{\partial u} - \frac{\partial F(x,u_0^+)}{\partial u} \right| dx$$

$$\leq \|u_n - u_0^+\|_\sigma \left\| \frac{\partial F(x,u_n)}{\partial u} \right\|_\gamma + \|u_0^+\|_\sigma \left\| \frac{\partial F(x,u_n)}{\partial u} - \frac{\partial F(x,u_0^+)}{\partial u} \right\|_\gamma$$

$\longrightarrow 0$, as $n \longrightarrow \infty$. Hence $\int_\Omega F(x,u_n)dx \longrightarrow \int_\Omega F(x,u_0^+)dx$ as $n \longrightarrow \infty$.
Now we aim to prove that $u_n \longrightarrow u_0^+$ strongly in E and $J_\lambda(u_0^+) = \theta_\lambda^+$.
Suppose otherwise then $\|u_0^+\| < \liminf_{n\longrightarrow\infty}(\|u_n\|)$ and so

$$\begin{aligned}
\langle J'(u_0^+), u_0^+ \rangle &= \|u_0^+\|^p - \int_\Omega F(x,u_0^+)dx - \lambda \int_\Omega a(x)|u_0^+|^q dx \\
&< \lim_{n\longrightarrow\infty} \inf \left(\|u_n\|^p - \int_\Omega F(x,u_n)dx - \lambda \int_\Omega a(x)|u_n|^q \right) dx \\
&< \lim_{n\longrightarrow\infty} \inf (0) = 0.
\end{aligned}$$

Thus $\|u_0^+\|^p - \int_\Omega F(x,u_0^+)dx - \lambda \int_\Omega a(x)|u_0^+|^q dx < 0$ but $u_0^+ \in N_\lambda$ a contradiction, therefore $u_n \longrightarrow u_0^+$ strongly. This implies $J_\lambda(u_n) \longrightarrow J_\lambda(u_0^+)$ as $n \longrightarrow \infty$.
To show that $J_\lambda(u_0^+) = \theta_\lambda$. By Fatous lemma and $u_0^+ \in N_\lambda(u)$ we get

$$\begin{aligned}
\theta_\lambda \leq J_\lambda(u_0^+) &= \frac{1}{p}\|u_0^+\|^p - \frac{1}{\sigma}\int_\Omega F(x,u_0^+)dx - \frac{\lambda}{q}\int_\Omega a(x)|u_0^+|^q dx \\
&\leq \liminf_{n\longrightarrow\infty}(\frac{1}{p}\|u_n\|^p - \frac{1}{\sigma}\int_\Omega F(x,u_n)dx - \frac{\lambda}{q}\int_\Omega a(x)|u_n|^q dx) \\
&\leq \liminf_{n\longrightarrow\infty} J_\lambda(u_n) = \theta_\lambda
\end{aligned}$$

$\theta_\lambda \leq J_\lambda(u_0^+)$ and $J_\lambda(u_0^+) \leq \theta_\lambda$ this implies $J_\lambda(u_0^+) = \theta_\lambda$. Finally we want to prove that u_0^+ is a nontrivial nonnegative solution of Equation (3.1) and $u_0^+ \in N_\lambda^+$

$$
\begin{aligned}
J_\lambda(u_n) &= \frac{1}{p}\|u_n\|^p - \frac{1}{\sigma}\int_\Omega F(x,u_n)dx - \frac{\lambda}{q}\int_\Omega a(x)|u_n^+|^q dx \\
&= \frac{1}{p}\|u_n\|^p - \frac{1}{\sigma}\left(\|u_n\|^p - \lambda\int_\Omega a(x)|u_n^+|^q dx\right) - \frac{\lambda}{q}\int_\Omega a(x)|u_n^+|^q dx \\
&= \frac{\sigma - p}{\sigma p}\|u_n\|^p - \lambda\frac{\sigma - q}{\sigma q}\int_\Omega a(x)|u_n^+|^q dx \\
&\geq -\lambda\frac{\sigma - q}{\sigma q}\int_\Omega a(x)|u_n^+|^q dx.
\end{aligned}
$$

By Theorem (3.1.17)(i) and lemma (3.1.11) $J_\lambda(u_n) \longrightarrow \theta_\lambda < 0$ as $n \longrightarrow \infty$, we obtain $\lambda \displaystyle\int_\Omega a(x)|u_0^+|^q dx > 0$. Thus u_0^+ is a nontrivial nonnegative.

Moreover, we have $u_0^+ \in N_\lambda^+$. If fact, if $u_0^+ \in N_\lambda^-$ then, there exist t_0^+, t_0^- such that $t_0^- u_0^+ \in N_\lambda^-$ and $t_0^+ u_0^+ \in N_\lambda^+$. In particular we have $t_0^+ < t_0^- = 1$. Since

$$
\frac{d^2}{dt^2}J_\lambda(t_0^+ u_0^+) > 0, \frac{d}{dt}J_\lambda(t_0^+ u_0^+) = 0,
$$

then there exist $t_0^+ < \tilde{t} < t_0^-$ such that $J_\lambda(t_0^+ u_0^+) < J_\lambda(\tilde{t} u_0^+)$. By lemma (3.1.14), we have

$$
J_\lambda(t_0^+ u_0^+) < J_\lambda(\tilde{t} u_0^+) \leq J_\lambda(t_0^- u_0^+) = J_\lambda(u_0^+) = \theta_\lambda,
$$

which is contradicts $J_\lambda(u_0^+) = \theta_\lambda^+$. Thus $u_0^+ \in N_\lambda^+$

Theorem 3.2.3:

If $0 < |\lambda| < \lambda_0$, then problem (3.1) has a positive solution u_0^- in N_λ^- such that

$$
J_\lambda(u_0^-) = \theta_\lambda^-.
$$

Proof. Similarly in the previous theorem since J_λ is bounded below on N_λ^-, then there exist a minimizing sequence u_n^- for J_λ on N_λ^- such that

$$
\begin{aligned}
J_\lambda(u_n) &= \theta_\lambda^- + o(1) \\
J_\lambda^{'}(u_n) &= o(1) \text{ in } E^*
\end{aligned}
$$

Again then there exist a subsequence u_n and $u_0^- \in N_\lambda^-$ is a nonzero solution of equation (3.1). Assume, without loss of generality, that

$$u_n \longrightarrow u_0^- \text{ weakly in } E, u_n \longrightarrow u_0^- \text{ strongly in } L^q, L^\sigma.$$

Moreover, let $u_n \in N_\lambda^-$, then by (3.21) we get

$$\int_\Omega F(x,u_n)dx > \frac{p-q}{\sigma-q}\|u_n\|^p, \tag{3.31}$$

So by (3.29) and (3.31) there exists a positive constant $\tilde{C}$ such that $\int_\Omega F(x,u_n)dx > \tilde{C}$. This implies

$$\int_\Omega F(x,u_0^-)dx \geq \tilde{C}, \tag{3.32}$$

Clearly by Lemma (3.1.12) and Equation (3.32) u_0^- is a nonnegative solution of Equation (3.1). Now, we aim to prove that $u_n \longrightarrow u_0^-$ strongly in $E, J_\lambda(u_0^-) = \theta_\lambda^-$ Supposing otherwise, then $\|u_0^-\| < \liminf\limits_{n\longrightarrow\infty} \|u_n\|$ and so by Lemma (3.1.14), then there exist a unique t_0^- such that $t_0^- u_0^- \in N_\lambda^-$. Since $u_n \in N_\lambda^-$, $J_\lambda(u_n) \geq J_\lambda(tu_n)$ $\forall$ $t \geq 0$, we have $J_\lambda(t_0^- u_0^-) < \lim\limits_{n\longrightarrow\infty} J_\lambda(t_0^- u_n) \leq \lim\limits_{n\longrightarrow\infty} J_\lambda(u_n) = \theta_\lambda^-$, which is a contradiction. Hence $u_n \longrightarrow u_0^-$ strongly in E, this imply that $J_\lambda(u_n) \longrightarrow J_\lambda(u_0^-) = \theta_\lambda^-$ as $n \longrightarrow \infty$.

Next we begin to show the proof of Theorem (3.2.1) in the following corollary.

Corollary 3.2.4:

Equation (3.1) has at least two positive solutions whenever $0 < |\lambda| < \lambda_0$.

Proof. BY Theorems (3.2.2) and (3.2.3) there exist $u_0^+ \in N_\lambda^+$ and $u_0^- \in N_\lambda^-$ such that $J_\lambda(u_0^+) = \inf\limits_{u\in N_\lambda^+} J_\lambda(u)$ and $J_\lambda(u_0^-) = \inf\limits_{u\in N_\lambda^-} J_\lambda(u)$. since $N_\lambda^+ \cap N_\lambda^- = \phi$, this implies that u_0^+ and u_0^- are distinct positive solution of Equation (3.1).

Chapter 4

p-Laplace Equation Model for Image Denoising

A well known inverse problem in image processing is image denoising which means the process with which we reconstruct a signal from a noisy one, by removing unwanted noise in order to restore the original image, or the method of estimating the unknown signal from available noisy data. The goal of image denoising is to remove the noise from the image but to preserve the useful information. Further image denoising is an important pre-processing step for image analysis. Let $u(x, y)$ denote the desired clean image, so $u_0 = u + n$, where n is the additive noise, u_0 denote the pixel values of a noisy image for $x, y \in \Omega$. Many authors has introduce algorithem to remove noise from images. In the last decades the energy functional approach together with its corresponding Euler Lagrange equation has attracted great attention in solving inverse problem applied to image reconstruction. One important case of Euler Lagrange equations is the one which involves the $p-$Laplace operator

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u); p \geq 1$$

associated with the evolution equation of $p-$Laplacian

$$\partial_t u = \Delta_p u, \text{ in } \Omega$$

$$u(0) = u_0, \text{ in } \Omega$$

$$\partial_N u = 0, \text{ on } \partial\Omega$$

where Ω is abounded domain in $\mathbb{R}^2$ and $u_0 : \Omega \longrightarrow \mathbb{R}$ is a given degraded image and ∇u is the gradient. It is well known that the case $p = 2$ gives the linear Gaussian filter, which however, impose strong spatial regularity and therefore image details such as lines and edges are over smoothed. The case $p = 1$ is often refereed to as the method of total variation and $p = 0$ is an instance of the so called balanced forward backward evolution. A $p-$Laplace equation model is proposed in this research for image denoising. First, the $p-$Dirichlet integral and total variation are combined to create a new energy functional used to built an image denoising model. This model is the generalization of Rudin-Osher-Fatemi model and Chambolle-Lions model. Generally, the practical images always hold the noise that does not only undermine the display but also affect the subsequent treatment results of the higher-level image. It is a big challenges to remove the noise of images with the maintainance of geometric characters during the scientific research and engineering practical activities. Therefore, denoising of image denoising is one of the important issues in the study of image processing and computer vision. Image denoising based on nonlinear diffusion equation is an effective method, about which many research achievements have been obtained and applied in many fields (Chan and Shen, 2005; Lysaker and Tai, 2006; Perona and Malik, 1990), see the references in [24]. The basic idea is to use different smooth policies at the target edge, namely at the edge area, the smooth process will be controlled but accelerated in the other regions. Based on the nonlinear diffusion equation, the complex filtering process can be divided into two simple ones: one along the image gradient direction and the other perpendicular to the image gradient direction. The equations with better denoising results should have various diffusion rates in both directions, namely, diffusion process is anisotropics. This method can also retain the image geometry while removing the noise. There are some classic and anisotropics diffusion models such as Perona-Malik model, mean curvature motion model, total variation model, among which total variation model (Rudin et al., 1992), [17], give the following energy functional equation:

$$E(u) = \int_{\Omega} |\nabla u| dx + \frac{\lambda}{2} \int_{\Omega} (u - u_0)^2 dx.$$

In the model, BV energy terms of the image Function model (based on the image gradient pattern energy term with L_1 norm determined) determine the corresponding evolution equation that has good non linear diffusion properties. In fact, the diffusion is unidirectional with non-zero diffusion velocity only in the tangent direction of horizontal lines of images which determines no demolishment of the important features of image structure but a certain effect of denoising during the evolution of the equation. However, in the local area of unimportant characteristics, the unidirectional diffusion speed becomes too slow and too single so as to affect the denoising effect and efficiency. The material through this chapter is mainly covered in [24], [6].

4.1 p-Laplacian for Image Denoising

An important feature in any evolution process for image denoising is preservation of certain geometrical features of the underlying image. In the case of image restoration these features include edges and corners. It is straight forward to express the $p-$laplace operator (1) as

$$\Delta_p u = |\nabla u|^{p-1}\Delta_1 u + (p-1)|\nabla u|^{p-2}\Delta_\infty u$$

where $\Delta_1 u = div(\frac{\nabla u}{|\nabla u|}),\ \Delta_\infty u = \frac{\nabla u}{|\nabla u|}D^2 u.\frac{\nabla u}{|\nabla u|}$ and $D^2 u$ is the Hessian of u. However, an intuitive way to represent Δ_p, giving direct interpretation of the diffusivity directions is to express Δ_p by using Gauge coordinates $(x, y) \longrightarrow (T, N)$:

$$\Delta_p u = u_N^{p-2}(u_{NN} + (p-1)u_{TT}).$$

Gauge coordinates

An image can be thought of as a collection of curves with equal value, the isophotes. At extrema an isophote reduces to a point, at saddle points the isophote is self-intersectin. At the non critical points Gauge coordinates (T, N)(or (v, w),or (ξ, η),or...) can be chosen. Gauge coordinates are locally set such that the T direction is tangent to the isophote and the N direction points in the direction of the gradient vector. Consequently, the unit vectors in the gradient and tangential direction are:

$$N = \frac{1}{\sqrt{u_x^2 + u_y^2}} \begin{pmatrix} u_x \\ u_y \end{pmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} N$$

as T perpendicular to N. The directional differential operator in the directions T and N are defined as

$$\partial_T = T.\nabla = T.(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \text{ and } \partial_N = N.\nabla = N.(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}).$$

Higher order derivatives are constructed through applying multiple first order derivatives, as many as needed. So u_{TT}, the second order derivative with respect to T is

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{1}{\sqrt{u_x^2 + u_y^2}} \begin{pmatrix} u_x \\ u_y \end{pmatrix} .(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \right)^2 u(x, y).$$

This implies that

$$u_T = \frac{\partial u}{\partial T} = \frac{u_y \partial_x u - u_x \partial_y u}{|\nabla u|} = 0 \text{ and } u_N = \frac{\partial u}{\partial N} = \frac{u_x \partial_x u + u_y \partial_y u}{|\nabla u|} = \frac{u_x^2 + u_y^2}{\sqrt{u_x^2 + u_y^2}} = |\nabla u|.$$

The second order structures are given as

$$\begin{aligned} u_{TT} &= \frac{u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy}}{u_x^2 + u_y^2} \\ u_{NN} &= \frac{u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy}}{u_x^2 + u_y^2}. \end{aligned}$$

These Gauge derivatives can be expressed as a product of gradients and the Hessian matrix H with second order derivatives:

$$u_{NN} u_N^2 = \nabla u H . \nabla^T u$$

$$u_{TT} u_N^2 = \nabla u H^r . \nabla^T u$$

with $\nabla u = (u_x, u_y)$, H is the Hessian matrix, and $H^r = detH \times H^{-1}$. Note that the expressions are invariant with respect to the spatial coordinates. Furthermore, one gets $\Delta u = u_{NN} + u_{TT}$. In gauge coordinates the cartesian formula for isophote curvature is easily derived by applying implicit differentiation twice.

The definition of an isophote is $u(T, N(T)) = c$, where c is a constant.

One time implicit differentiation with respect to T gives:

$$u_T + u_{N(T)}N'(T) = 0,$$

from which follow that $N'(T) = 0$ because $u_T = 0$ by definition. Using that and second implicit differentiation gives:

$$u_{TT} + 2u_{TN}N'(T) + u_{NN}(N'(T))^2 + u_N N''(T) = 0.$$

The isophote curvature k is defined as $N''(T)$, the change of the tangent vector $N'(T)$ in the T direction, so

$$k = N''(T) = \frac{-u_{TT}}{u_N} = \frac{u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}}{(u_x^2 + u_y^2)^{\frac{3}{2}}}.$$

Minimizing methods

Consider an image u on the domain Ω, the first variation of the functional E at u in the direction v is defined by

$$\delta E(u, v) = \frac{d}{d\epsilon}E(u + \epsilon v)|_{\epsilon=0}.$$

The variational derivative $\delta E(u)$ of the functional E at u in the direction v is defined by

$$\delta E(u, v) = \int_\Omega \delta E(u).v dx$$

with $v \in C_0^\infty(\Omega)$ a test function that is zero at the boundaries. Minimizing u with appropriate boundary conditions gives the Euler Lagrange equation $\delta E = 0$. Adynamical system is obtained by the steepest decent approach $u_t = -\delta E$. So to find the minimum of $E(u)$ given an image u_0 is to solve

$$\begin{aligned} u_t &= -\delta E(u) \\ u(0) &= u_0. \end{aligned}$$

For $p-$Laplacians we consider in general the integral $E(u) = \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p d\Omega$. It is well known as the $p-$Dirichlet energy integral with a companying $p-$Laplacian equation $\delta E = 0$, with $\delta E = -\nabla.(|\nabla u|^{p-2}\nabla u)$. Using gauge coordinates the energy can be written as $E_p(u) = \dfrac{1}{p}\displaystyle\int_\Omega u_N^p dx$.

Theorem 4.1.1:

The variational derivative $\delta E(u)$ can be written as

$$\delta E(u) = -u_N^{p-2}(u_{TT} + (p-1)u_{NN}).$$

Proof.

$$\begin{aligned}
\delta E(u, v) &= \frac{1}{p}\int_\Omega \frac{d}{d\epsilon}|\nabla(u+\epsilon v)|^p dx|_{\epsilon=0} \\
&= \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx \\
&= |\nabla u|^{p-2}\nabla u.v|_{\partial\Omega} - \int_\Omega \nabla.(|\nabla u|^{p-2}\nabla u)v dx,
\end{aligned}$$

since $v = 0$ on the boundary, $|\nabla u|^{p-2}\nabla u.|_{\partial\Omega} = 0$ and the Euler Lagrange equation $\delta E(u) = 0$ equals

$$-(\nabla.(|\nabla u|^{p-2}\nabla u)) = 0.$$

The left hand side equals the well known variational derivative of the Laplacian.

An explicit expressions is obtained by applying the divergence operator to both terms, where Gauge coordinates are used:

$$\begin{aligned}
-\nabla.(|\nabla u|^{p-2}\nabla u) &= -\nabla(|\nabla u|^{p-2}).\nabla u - |\nabla u|^{p-2}(\nabla.\nabla u) \\
&= -(\nabla.u_N^{p-2}).\nabla u - u_N^{p-2}\Delta u.
\end{aligned}$$

For the first part we have

$$\begin{aligned}
(\nabla.u_N^{p-2}).\nabla u &= (p-2)u_N^{p-3}\nabla u_N.\nabla u \\
&= (p-2)u_N^{p-3}(\nabla u H u_N^{-1}).\nabla u,
\end{aligned}$$

where H is the Hessian matrix. Recall $(\nabla u H).\nabla u = u_N^2 u_{NN}$ as given before. Therefore

$$(p-2)u_N^{p-3}u_N^{-1}u_N^2 u_{NN} = (p-2)u_N^{p-2}u_{NN}$$

and consequently we have

$$\delta E(u) = -\left((p-2)u_N^{p-2}u_{NN} + u_N^{p-2}\Delta u\right).$$

Using the identity $\Delta u = u_{NN} + u_{TT}$ this gives

$$\delta E(u) = -u_N^{p-2}\left(u_{TT} + (p-1)u_{NN}\right).$$

For $p = 2$ we have the heat equation:

$$u_N^{2-2}(u_{TT} + (2-1)u_{NN}) = u_{TT} + u_{NN} = \Delta u.$$

Next, $p = 1$ gives the Total variation flow:

$$u_N^{1-2}(u_{TT} + (1-1)u_{NN}) = u_N^{-1}u_{TT} = K.$$

In general, it gives a recipe for PDE-driven flow:

$$u_t = u_N^{p-2}(u_{TT} + (p-1)u_{NN}).$$

The case $p \longrightarrow \infty$ is known as the infinite Laplacian, denoted by $\Delta_\infty u$. This term is defined as either u_{NN} or $u_N^2 u_{NN}$. It can be applied to image inpainting and shape metamorphism.

4.2 Model of Image Denoising Based on the p-Laplace Equation

Chambolle and Lions use the heat diffusion term to accelerate the total variation model partially (Chambolle, 1995), [3]. Chen et al. (2006), [25], studied the diffusion behaviours of variational exponentiate. With the inspiration of these studies,(Wei, Wei and Bin Zhou,(2012)), [24], proposed the following functional to build a model used in image denoising

$$E(u) = \int_\Omega F(|\nabla u|)\, dx + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 dx$$

where, u_0 refers to the images that is needed to be denoised and the nonnegative function $F(s)$ is defined by

$$F(s) = \begin{cases} \frac{1}{p}s^p, & 0 \le s \le \beta, \\ \beta^{p-1}s + (1-\frac{1}{p})\beta^p, & s > \beta. \end{cases}$$

Let M denote the manifold of smooth images, then the diffusion equations presented can be interpreted as the gradient decent equations for the minimization of the energy functional $E : M \longrightarrow \mathbb{R}$ defined by:

$$E(u) = \int_\Omega F(|\nabla u|)\, dx + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 dx.$$

Then for any function $v \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}
\delta E(u,v) &= \frac{d}{d\epsilon}\left(\int_\Omega F(|\nabla(u+\epsilon v)|)dx + \frac{\lambda}{2}\int_\Omega (u+\epsilon v-u_0)^2 dx\right)|_{\epsilon=0} \\
&= \int_\Omega \frac{d}{d\epsilon}F(|\nabla u+\epsilon\nabla v|)dx|_{\epsilon=0} + \frac{\lambda}{2}\int_\Omega \frac{d}{d\epsilon}(u+\epsilon v-u_0)^2 dx|_{\epsilon=0} \\
&= \int_\Omega F'(|\nabla u|)|\nabla u|^{-1}\nabla u.\nabla v + \lambda\int_\Omega (u-u_0)vdx \\
&= F'(|\nabla u|)|\nabla u|^{-1}\nabla u.v|_{\partial\Omega} - \int_\Omega \nabla.\left(F'(|\nabla u|)|\nabla u|^{-1}\nabla u\right)vdx + \lambda\int_\Omega (u-u_0)vdx \\
&= -\int_\Omega \left[\nabla.\left(F'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) - \lambda(u-u_0)\right].vdx.
\end{aligned}$$

Hence the Euler Lagrange equation $\delta E(u) = 0$ reads $\left[\nabla.\left(F'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) - \lambda(u-u_0)\right] = 0$. Since $\frac{\partial u}{\partial t} = -\delta E(u)$, then the following evolution equation is obtained

$$\frac{\partial u}{\partial t} = \nabla.\left(F'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) - \lambda(u-u_0) \tag{4.1}$$

where

$$\nabla.\left(F'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = \begin{cases} \nabla.(|\nabla u|^{p-2}\nabla u), & 0 \le |\nabla u| \le \beta, \\ \beta^{p-1}\nabla.\frac{\nabla u}{|\nabla u|}, & |\nabla u| > \beta. \end{cases}$$

Theorem 4.2.1:

The Eq (4.1) is equal to the following form:

$$\frac{\partial u}{\partial t} = \begin{cases} u_N^{p-2}(u_{TT} + (p-1)u_{NN}) - \lambda(u - u_0), \ 0 < u_N \leq \beta, \\ \dfrac{\beta^{p-1}}{u_N} u_{TT} - \lambda(u - u_0), \ \ u_N > \beta. \end{cases}$$

Proof. From Equation (4.1) we have

$$\frac{\partial u}{\partial t} = \begin{cases} \nabla.(|\nabla u|^{p-2}\nabla u) - \lambda(u - u_0), \ \ 0 \leq |\nabla u| \leq \beta, \\ \beta^{p-1}\nabla.\dfrac{\nabla u}{|\nabla u|}, \ \ |\nabla u| > \beta, \end{cases}$$

when $0 \leq |\nabla u| \leq \beta$

$$\frac{\partial u}{\partial t} = \nabla.\left(|\nabla u|^{p-2}\nabla u\right) - \lambda\left(u - u_0\right).$$

using gauge coordinates we get

$$\begin{aligned} \nabla.(|\nabla u|^{p-2}\nabla u) &= \nabla(|\nabla u|^{p-2}).\nabla u + |\nabla u|^{p-2}(\nabla.\nabla u) \\ &= (\nabla.u_N^{p-2}).\nabla u + u_N^{p-2}\Delta u. \end{aligned}$$

For the first part we have

$$\begin{aligned} \nabla(u_N^{p-2}).\nabla u &= (p-2)u_N^{p-3}\nabla u_N.\nabla u \\ &= (p-2)u_N^{p-3}(\nabla u H u_N^{-1}).\nabla u, \end{aligned}$$

where H is the Hessian matrix. Recall that $\nabla u H.\nabla u = u_N^2 u_{NN}$, thus

$$(p-2)u_N^{p-3}u_N^{-1}u_N^2 u_{NN} = (p-2)u_N^{p-2}u_{NN}.$$

Therefore

$$\frac{\partial u}{\partial t} = (p-2)u_N^{p-2}u_{NN} + u_N^{p-2}\Delta u - \lambda(u - u_0).$$

Using the identity $\Delta u = u_{NN} + u_{TT}$, we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} &= (p-2)u_N^{p-2}u_{NN} + u_N^{p-2}(u_{NN} + u_{TT}) - \lambda(u - u_0) \\ &= u_N^{p-2}\left((p-2)u_{NN} + u_{NN} + u_{TT}\right) - \lambda(u - u_0) \\ &= u_N^{p-2}(u_{TT} + (p-1)u_{NN}) - \lambda(u - u_0), \end{aligned}$$

when $|\nabla u| > \beta$

$\dfrac{\partial u}{\partial t} = \beta^{p-1}\nabla.\dfrac{\nabla u}{|\nabla u|} - \lambda(u-u_0)$, but $\nabla.\dfrac{\nabla u}{|\nabla u|} = k = u_N^{-1}u_{TT}$. Thus

$$\frac{\partial u}{\partial t} = \beta^{p-1}\frac{u_{TT}}{u_N} - \lambda(u-u_0)$$

4.3 Numerical Experiment

In this section the numerical experiments with different parameters are implemented by using the above mentioned model, we propose the preliminary boundary problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla.\left(F'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) - \lambda(u-u_0), \quad \text{in } \Omega \\ \frac{\partial u}{\partial N}(x,t) &= 0, \quad \text{on } \partial\Omega \\ u(x,0) &= u_0(x), \quad \text{in } \Omega \end{aligned} \tag{4.2}$$

where, the definition of $\nabla.\left(F'(|(\nabla u|)\dfrac{\nabla u}{|\nabla u|}\right)$ is similar to that in Equation (4.1). Using the method of [17], both sides of the first formula are multiplied by $u-u_0$ and integration over Ω is performed. Since $t \longrightarrow \infty$, $\dfrac{\partial u}{\partial t} \longrightarrow 0$, then

$$\int_\Omega \nabla.\left(F'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)(u-u_0)dx = \int_\Omega \lambda(u-u_0)^2dx.$$

Using the Green formula we obtain

$$\int_{\partial\Omega} \frac{F'(|\nabla u|)}{|\nabla u|}\frac{\partial u}{\partial N}(u-u_0)ds - \int_\Omega \frac{F'|\nabla u|}{|\nabla u|}\nabla u.\nabla(u-u_0)dx = \lambda\int_\Omega (u-u_0)^2dx.$$

Since $\dfrac{\partial u}{\partial N}|_{\partial\Omega} = 0$, then $\displaystyle\int_{\partial\Omega} \frac{F'(|\nabla u|)}{|\nabla u|}\frac{\partial u}{\partial N}(u-u_0)ds = 0$. Hence

$$\lambda = \frac{-\displaystyle\int_\Omega F'(|\nabla u|)\frac{\nabla u}{|\nabla u|}.\nabla(u-u_0)dx}{\displaystyle\int_\Omega (u-u_0)^2dx}.$$

The problem (4.2) is solved Numerically and then it could be used in the field of denoising image. As shown in Fig. 4.1(a) and (b), given original images of phoenix tree leaves and denoised images, different p values are chosen to perform numerical solution which produces corresponding results. As a second test, Fig 4.2 (a) and (b) dividedly the original rice-grains images and denoising images. The result of various p values and shown in Fig 4.1 (c-f) and Fig 4.2 (c-f) show the different results in the two experiments with various $p-$values. The $p-$values, iterative numbers n of model solution, the solved results of Peak-Signal-to-Noise Ratio ($PSNR$) are indicated in Table 4.1 and 4.2.

Table 4.1: p and n and $PSNR$ data

p	n	$PSNR$
1.0	430	24.7805
1.6	90	26.2181
2.0	27	26.0873
2.2	14	25.8030

Table 4.2: Specific data of p, n and $PSNR$

p	n	$PSNR$
1.0	364	22.5570
1.6	65	22.6794
2.0	17	22.1307
2.2	9	21.7861

Given the constant $p-$value, with the iterative evolution, $PSNR$ is gradually increased from values of 18.9763 and 18.7481 to the final results. The process is stable.

When, n is decreased by $p-$value, the $PSNR$ value of final results is changed with a tiny visual effection. Notice that is defined as follows

$$PSNR \ = 10\log_{10}\left(\frac{max(x_{i,j})^2}{\text{MSE}}\right).$$

Where MSE=(Mean Squar Error) is given by

$$MSE \ = \frac{\sum_{i,j}(x_{i,j}-y_{i,j})^2}{MN}$$

with M, N are the total number of pixels in the horizontal and vertical dimensions of the image, $x_{i,j}$ and $y_{i,j}$ denote the original and distorted image, respectively.

Conclusion

In the current work we introduced method to solve elliptic partial differential equation with homogeneous Dirichlet boundary conditions called Nehari method. We use this method to prove that the $p-$Laplace equation with Dirichlet boundary condition has at least two positive solutions. Further in this study we apply $p-$Laplace equation in denoising process of images. The test results show that, according to the reasonably adjusting parameter p values, the iterative numbers decrease with better denoising effects.

References

[1] A Bressan, https://pdfs.semanticscholar.org/646c/6e5adf1715fba102c62f7ca9ad22bc52f843.pdf

[2] Andrzej Szulkin and Tobias Weth. The method of Nehari manifold. Handbook of nonconvex analysis and applications, 597632, 2010.

[3] Antonin Chambolle. Image segmentation by variational methods: Mumford and shah functional and the discrete approximations. SIAM Journal on Applied Mathematics, 55(3):827863, 1995.

[4] Antonio Ambrosetti, Ham Brezis, and Giovanna Cerami. Combined effects of concave and convex nonlinearities in some elliptic problems. Journal of Functional Analysis, 122(2):519543, 1994.

[5] Arjan Kuijper. Geometrical pdes based on second-order derivatives of gauge coordinates in image processing. Image and Vision Computing, 27(8):1023 1034, 2009.

[6] Arjan Kuijper. $p-$Laplacian driven image processing. In Image Processing, 2007. ICIP 2007. IEEE International Conference on, volume 5, pages V 257. IEEE, 2007.50

[7] C Atkinson and K El-Ali. Some boundary value problems for the bingham model. Journal of non-newtonian fluid mechanics, 41(3):339363, 1992.

[8] Francine Catte, Pierre-Louis Lions, Jean-Michel Morel, and Tomeu Coll. Image selective smoothing and edge detection by nonlinear diffusion. SIAM Journal on Numerical analysis, 29(1):182193, 1992.

[9] Ghanmi Abdeljabbar. Existence of nontrivial solutions of $p-$Laplacian equation with sign-changing weight functions. ISRN Mathematical Analysis, 2014, 2014.

[10] Ham Brezis and Louis Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Communications on Pure and Applied Mathematics, 36(4):437477, 1983.

[11] H. Amann and J. Lopez-Gomez. A priori bounds and multiple solutions for superlinear indefinite elliptic problems. Journal of Differential Equations, 146(2):336374, 1998.

[12] Hossein T Tehrani. A multiplicity result for the jumping nonlinearity problem. Journal of Differential Equations, 188(1):272305, 2003.

[13] Jan J Koenderink. The structure of images. Biological cybernetics, 50(5):363370, 1984.

[14] Juan Luis Vazquez. Smoothing and decay estimates for nonlinear diffusion equations: equations of porous medium type, volume 33. Oxford University Press, 2006.

[15] Khaled Ben Ali and Abdeljabbar Ghanmi. Nehari manifold and multiplicity result for elliptic equation involving p-Laplacian problems. Boletim da Sociedade Paranaense de Matematica, 36(4):197208, 2018.

[16] KJ Brown. The Nehari manifold for a semilinear elliptic equation involving a sublinear term. Calculus of variations and partial differential equations, 22(4):483494, 2004.

[17] Leonid I Rudin, Stanley Osher, and Emad Fatemi. Nonlinear total variation based noise removal algorithms. Physica D: Nonlinear Phenomena, 60(1- 4):259268, 1992.

[18] Michel Willem. Minimax theorems, volume 24. Springer Science Business Media, 1997.

[19] Paul A Binding, Pavel Drabek, and Yin Xi Huang. On Neumann boundary value problems for some quasilinear elliptic equations. Electronic Journal of Differential Equations, 1997(05):111, 1997.

[20] Pavel Drabek and Stanislav I Pohozaev. Positive solutions for the $p-$Laplacian: application of the fibrering method. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 127(4):703726, 1997.

[21] P Lindqvist, https://folk.ntnu.no/lqvist/p-laplace.pdf

[22] Robert A Adams and John JF Fournier. Sobolev spaces, volume 140. Academic press, 2003.

[23] Tsung-Fang Wu. On semilinear elliptic equations involving concave convex nonlinearities and sign-changing weight function. Journal of Mathematical Analysis and Applications, 318(1):253270, 2006.

[24] Wei Wei and Bin Zhou. A $p-$Laplace equation model for image denoising. Information Technology Journal, 11(5):632, 2012.

[25] Yunmei Chen, Stacey Levine, and Murali Rao. Variable exponent, linear growth functionals in image restoration. SIAM journal on Applied Mathematics, 66(4):13831406, 2006.

[26] Yuri Bozhkov and Enzo Mitidieri. Existence of multiple solutions for quasilinear systems via fibering method. Journal of Differential Equations,190(1):239267, 2003.

Figures

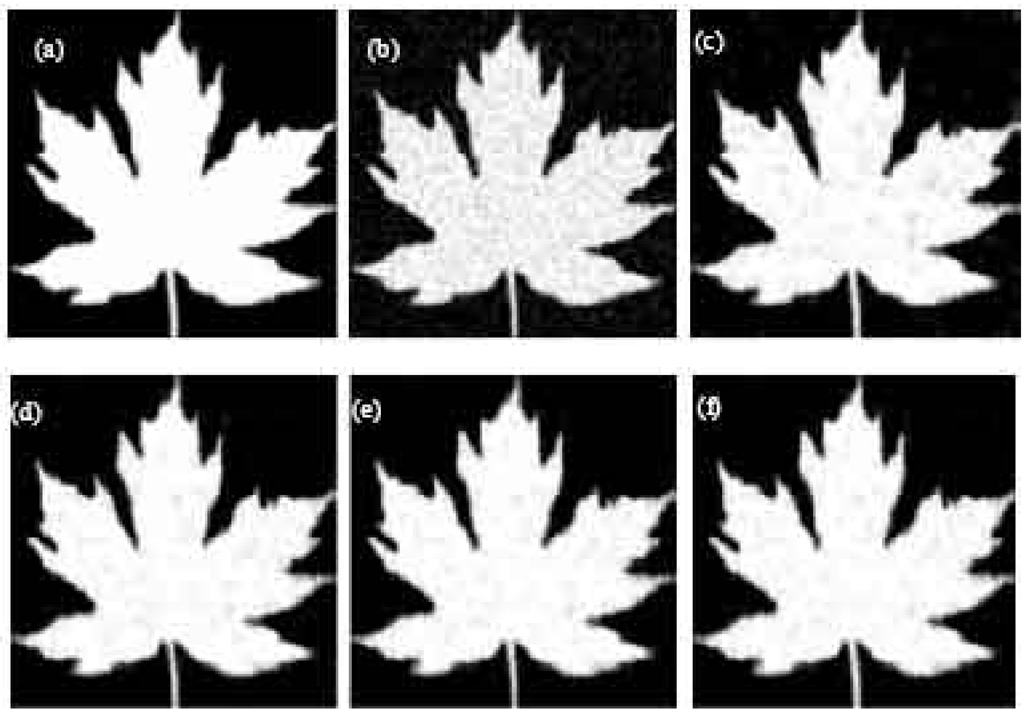

Figure 4.1: **(a-f): Results of transaction of phoenix tree leaves with noise; (a) Original image, (b) Noise image, (c) p = 1.0, (d) p = 1.6, (e) p = 2.0 and (f) p = 2.2 .**

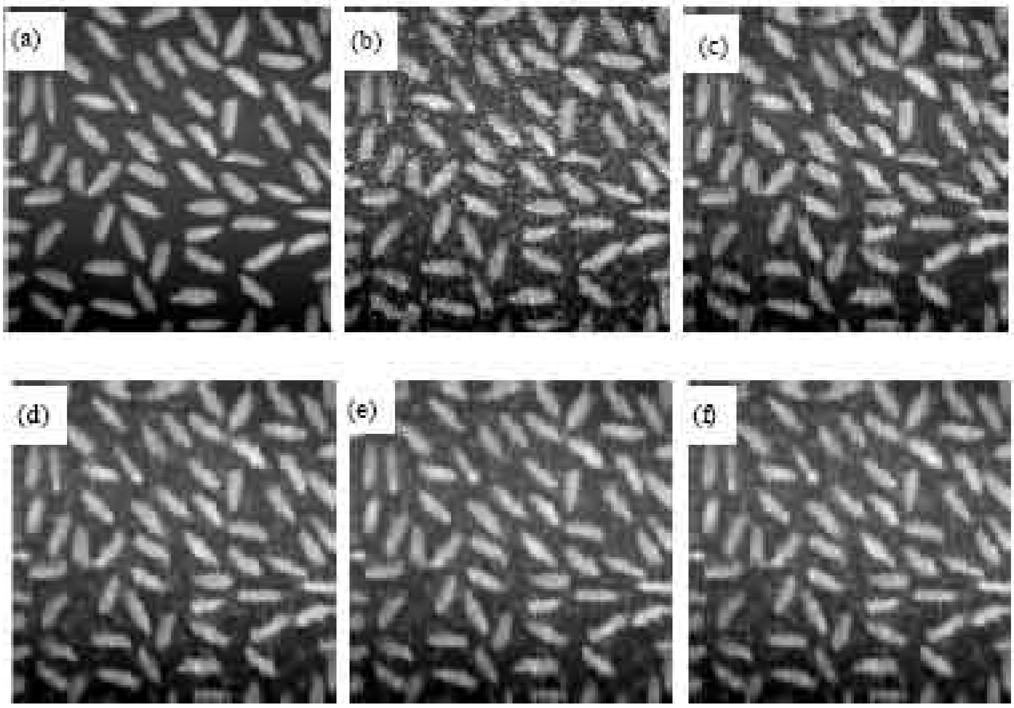

Figure 4.2: **(a-f): Results of rice-grains with noise; (a) Original image, (b) noise image, (c) p = 1.0, (d) p = 1.6, (e) p = 2.0 and (f) p = 2.2 .**

حول حلول معادلات p لابلاس

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إشراف: الدكتور يوسف زحايقه

ملخص:

تلعب معادلات p لابلاس دورا مهما في توصيف العديد من الظواهر الفيزيائية. في هذه الاطروحه درسنا المشكلة المحيطية والمكونة من معادله p لابلاس وشرط دريشليت المحيطي والمعطاة وفق الصورة التالية:

$$\Delta_p u = \frac{1}{\sigma}\frac{\partial F(x,u)}{\partial u} + \lambda a(x)|u|^{q-2}u, \qquad in\ \Omega \qquad (1)$$

$$u = 0, \qquad on\ \partial\Omega.$$

حيث ان المجال Ω هو مجموعه جزئيه من R^{n}, $n \geq 3$ مفتوحه ومحدوده والمعاملات $q\ ,p\ ,\sigma\ ,p^*$ تحقق $1 < q < p < \sigma < p^*$ و λ هو عدد حقيقي ماعدا الصفر والاقتران $a(x)$ اقتران املس يمكن ان يغير من اشارته في المجال Ω . بينما F ينتمي الى الفضاء الاقتراني $C^1(\overline{\Omega} \times R, R)$ وهو عبارة عن اقتران متجانس بدرجة σ .

بشكل عام إيجاد حلول تحليليه (كلاسيكيه) لمعادلة p لابلاس امر غير يسير وعليه يتم ايجاد حلول ليست كلاسيكيه تستنبط بناء على طرق تعرف بطرق التغيرات المكافئه لمعادلة p لابلاس والشرط المحيطي المرادف لها. في هذه الاطروحه تم الاعتماد على طريقة نيهاري في برهنة نتيجة اساسيه وهي وجود اكثر من حل موجب للنظام (1) ينتمي الى متعدد طبقات نيهاري الذي هو جزء من فضاء سوبولوف $W_0^{1,p}$. كذلك في هذه الاطروحه تم تطبيق معادلة p لابلاس في مجال تنقية الصور المشوشه حيث تلعب المعادلات التفاضليه الجزئيه دورا اساسيا. بعض تلك المعادلات تنشأ عن تصغير اقترانات الطاقه والطرق التي تستخدمها تسمى طرق التغيرات التامه ،[17]. بعض الطرق الاخرى تعتمد معادلات تصمم بتبرير هندسي كمماسات منحنيات تساوي الصورة وهذه الطرق تعرف بطرق حركة مقياس الانحناء الوسطي ، [8]. في هذه الاطروحه تم اعتماد طريقه تمزج بين الطريقتين تقوم على حل معادله تفاضليه (معادلة انتشار) تحوي ال p لابلاس اوبريتر ,[24].

وقد تم تطبيق هذه الطريقه من خلال مثال عددي تم عرضه في هذه الاطروحه .